Kinetic models of elastic waves in bounded media

with applications in structural acoustics

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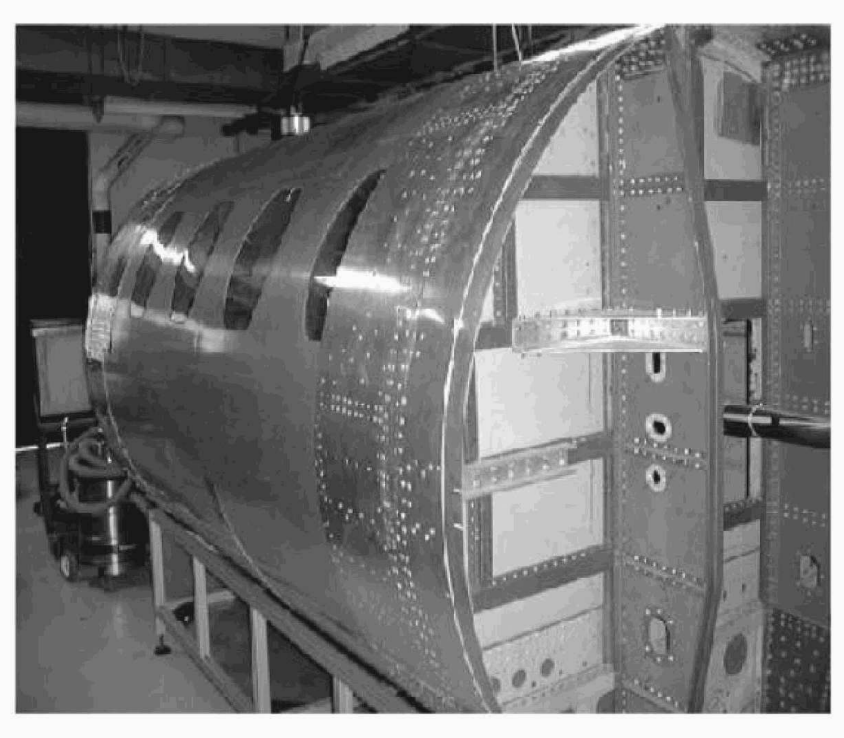
Outline

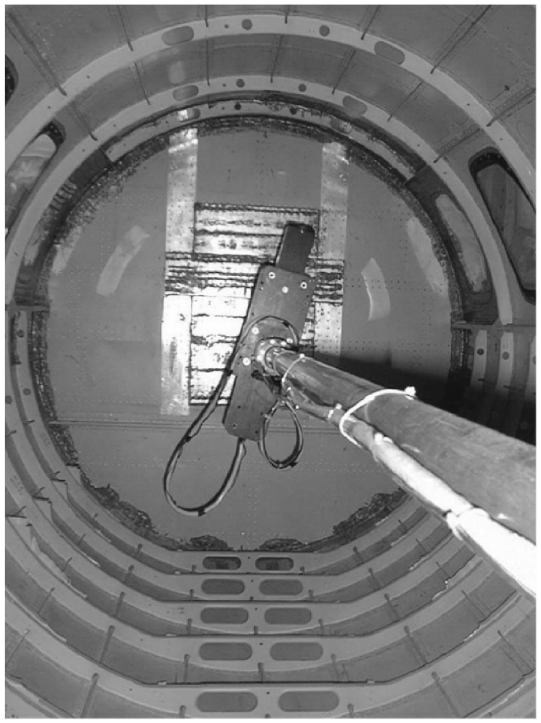
Outline

Frequency response function of complex structures

Example

- Cessna Citation fuselage: 9 floor/19 ceiling ribs, 22 stringers.
- Length = 2.55 m, radius = 0.81 m, thickness = 0.8-1.2 mm.

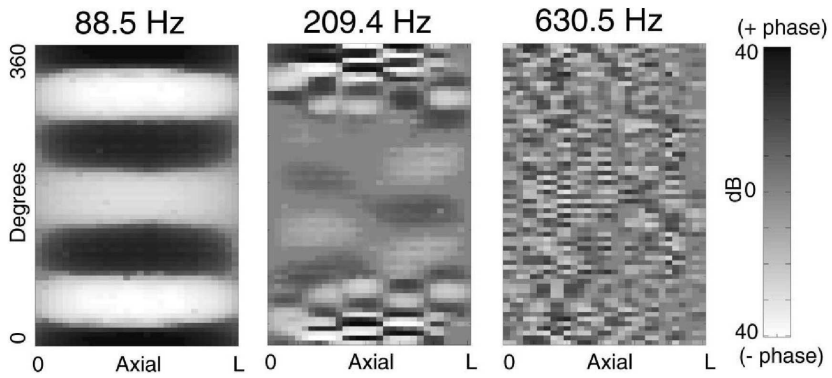




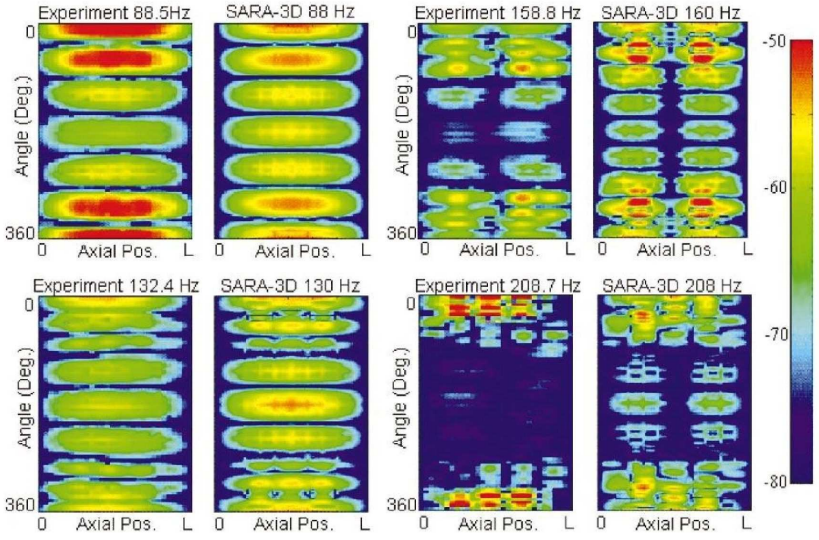
Herdic *et al.* JASA 117(6) 3667, 2005

Frequency response function of complex structures

Example



Phase distributions of surface velocity



Magnitude distributions of surface velocity

Excitation: point force at a rib/stringer stiffener intersection.

Structural-acoustics system

- Hypotheses: linear, visco-elastic materials with density $\rho(\mathbf{x})$ and relaxation tensor $\mathrm{C}(\mathbf{x}, t)$, $\mathbf{x} \in \mathcal{O} \subseteq \mathbb{R}^d$, $t \in \mathbb{R}_+$.
- Local balance of momentum for the displacement field $\mathbf{u}$ in $\mathcal{O} \times \mathbb{R}$:

$$\rho \partial_t^2 \mathbf{u} = \mathbf{Div}\, \boldsymbol{\sigma} \,.$$

- Constitutive equation for the stress field $\boldsymbol{\sigma}$ in $\overline{\mathcal{O}} \times \mathbb{R}_+$ as a function of the strain field $\boldsymbol{\epsilon}(\mathbf{u}) = \boldsymbol{\nabla}_{\mathbf{x}} \otimes_s \mathbf{u}$:

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \mathrm{C}(\mathbf{x}, 0)\boldsymbol{\epsilon}(\mathbf{u}) + \underbrace{\partial_t \mathrm{C}(\mathbf{x}, \cdot) *_t \boldsymbol{\epsilon}(\mathbf{u})}_{\text{ineffective in the high-frequency limit}} \,.$$

"High-frequency" setting 1/2

- High frequencies correspond to $\varepsilon \to 0$ for strongly "ε–oscillatory" initial conditions:

$$\mathbf{u}_\varepsilon(\mathbf{x},0) = \mathbf{u}_\varepsilon^0(\mathbf{x})\,, \quad \||\varepsilon\boldsymbol{\nabla}_\mathbf{x}|\mathbf{u}_\varepsilon^0\|_{L^2_{\rm loc}} < \infty\,,$$

  and:

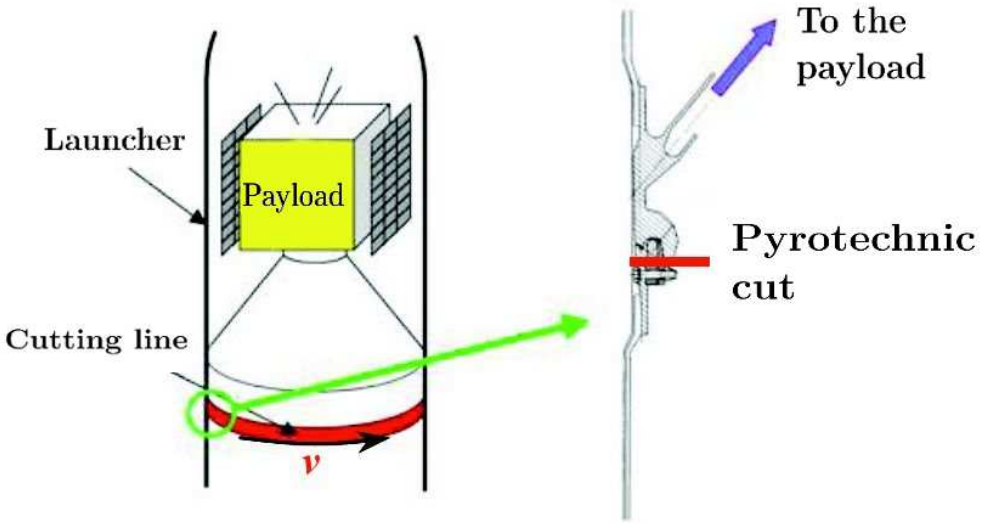
$$\partial_t\mathbf{u}_\varepsilon(\mathbf{x},0) = \mathbf{v}_\varepsilon^0(\mathbf{x})\,, \quad \||\varepsilon\boldsymbol{\nabla}_\mathbf{x}|\mathbf{v}_\varepsilon^0\|_{L^2_{\rm loc}} < \infty\,.$$

- **Example**: plane waves, $\varepsilon \equiv (|\mathbf{k}|L)^{-1}$,

$$\mathbf{u}_\varepsilon^0(\mathbf{x}) = \varepsilon\mathbf{A}(\mathbf{x})\,\mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}\mathbf{k}\cdot\mathbf{x}}\,, \quad \mathbf{v}_\varepsilon^0(\mathbf{x}) = \mathbf{B}(\mathbf{x})\,\mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}\mathbf{k}\cdot\mathbf{x}}\,.$$

Example: impulse loads in spatial structures

- Example #1: pyrotechnic cut.



Impulse loads $\longrightarrow$ *High frequency (HF) wave propagation.*

Example: impulse loads in spatial structures

- Example #2: solar panel unfolding.

Impulse loads $\longrightarrow$ *High frequency (HF) wave propagation.*

"High-frequency" setting 2/2

- Introduce the acoustic (Christoffel) tensor $\mathbf{\Gamma}$ of the medium:

$$\mathbf{\Gamma}(\mathbf{x},\mathbf{k})\mathbf{U} := \varrho(\mathbf{x})^{-1}(\mathrm{C}(\mathbf{x}) : \mathbf{U}\otimes\mathbf{k})\mathbf{k}\,, \quad \mathbf{k}\in\mathbb{R}^{d}\,,\ \mathbf{U}\in\mathbb{R}^{n}\,.$$

- The local balance of momentum for the displacement field $\mathbf{u}_\varepsilon$ in $\mathcal{O}\times\mathbb{R}_t$ is the elastic wave equation:

$$\varrho(\mathbf{x})\left(\mathbf{\Gamma}(\mathbf{x},\mathrm{i}\varepsilon\boldsymbol{\nabla}_{\mathbf{x}}) - (\mathrm{i}\varepsilon\partial_t)^2\right)\mathbf{u}_\varepsilon = \mathrm{O}(\varepsilon)\,,$$

  supplemented with (*e.g.* Neumann or Dirichlet) boundary conditions on $\partial\mathcal{O}\times\mathbb{R}_t$.

Some properties of $\boldsymbol{\Gamma}$

- $\boldsymbol{\Gamma}$ is symmetric, real, positive definite (in $\mathcal{O} \times \mathbb{R}^d \setminus \{\mathbf{k} = \mathbf{0}\}$): it as n (possibly r_α-multiple) positive eigenvalues λ_α^2 such that

$$\lambda_\alpha(\mathbf{x}, \mathbf{k}) = c_\alpha(\mathbf{x}, \hat{\mathbf{k}})|\mathbf{k}|\,, \quad 1 \leq \alpha \leq n\,,$$

where $\hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|} \in \mathbb{S}^{d-1}$, and the real eigenvectors $\mathbf{b}_\alpha(\mathbf{x}, \mathbf{k}) = \mathbf{b}_\alpha(\mathbf{x}, \hat{\mathbf{k}})$ can be orthonormalized,

$$\boldsymbol{\Gamma} = \sum_{\alpha=1}^{n} \lambda_\alpha^2 \mathbf{b}_\alpha \otimes \mathbf{b}_\alpha\,, \quad \mathbb{I}_n = \sum_{\alpha=1}^{n} \mathbf{b}_\alpha \otimes \mathbf{b}_\alpha\,.$$

- **Example**: three-dimensional isotropic medium ($n = d$),

$$\boldsymbol{\Gamma}(\mathbf{x}, \hat{\mathbf{k}}) = c_{\mathrm{P}}^2(\mathbf{x})\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} + c_{\mathrm{S}}^2(\mathbf{x})(\mathbb{I}_d - \hat{\mathbf{k}} \otimes \hat{\mathbf{k}})\,.$$

Energy and power flow densities

- The energy density (real, positive) and the power flow density (vector) are:

$$\mathcal{E}_\varepsilon(\mathbf{x},t) = \frac{1}{2}(\rho|\partial_t \mathbf{u}_\varepsilon|^2 + \boldsymbol{\sigma}_\varepsilon : \boldsymbol{\epsilon}_\varepsilon) = \mathcal{T}_\varepsilon + \mathcal{U}_\varepsilon \,,$$
$$\boldsymbol{\pi}_\varepsilon(\mathbf{x},t) = -\boldsymbol{\sigma}_\varepsilon \partial_t \mathbf{u}_\varepsilon \,.$$

- They satisfy for all *fixed* $\varepsilon \in]0, \varepsilon_0]$ a conservation equation:

$$\boxed{\partial_t \mathcal{E}_\varepsilon + \operatorname{div} \boldsymbol{\pi}_\varepsilon \; (+\pi_{\mathrm{dis}}) = 0 \; (+\pi_{\mathrm{inj}})} \,.$$

- What happens when $\varepsilon \to 0$?

Existing approaches for structural-acoustics

Statistical Energy Analysis (SEA)

Westphal 1957, Lyon-Maidanik 1962, Smith 1962, Maidanik 1976 ...

Vibrational conductivity analogy (VCA)

Rybak 1972, Nefske-Sung 1989, Belyaev 1991, Bernhard *et al.* 1992, Langley 1995, Ichchou-Jézéquel 1996, Vlahopoulos *et al.* 1999, Le Bot 2002.

Existing approaches for structural-acoustics

Ray methods with real phases (WKBJ/FIO)

Carlini 1817, Green 1837, Liouville 1837 [...] Keller 1958, Steele 1969, Pierce 1970, Germogenova 1973, Maslov-Fedoruk 1981, Norris 1995 ...

Ray methods with complex phases (Gaussian beams)

Babich 1968, Katchalov-Popov 1981, Červený *et al.* 1982, Ralston 1982, Norris 1988, Tanushev *et al.* 2007 ...

Kinetic models (applications in geophysics and seismology)

Weaver *et al.* 1990, Campillo-Fink-van Tiggelen *et al.* 1998, Sato-Fehler 1998, Papanicolaou-Ryzhik 1999, Scales *et al.* 2000.

Outline

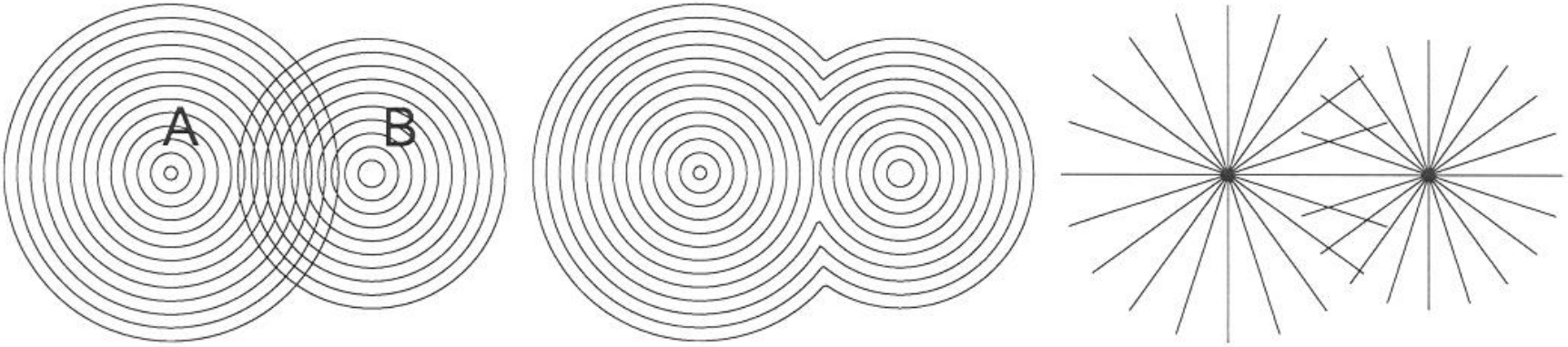
Ray methods 1/3

Principles

- $\mathbf{u}_\varepsilon$ is sought for as a WKBJ ansatz:

$$\mathbf{u}_\varepsilon(\mathbf{x}, t) \simeq e^{\frac{\mathrm{i}}{\varepsilon} S(\mathbf{x},t)} \sum_{k=0}^{\infty} \varepsilon^k \mathbf{U}_k(\mathbf{x}, t)\,.$$

- **Eulerian** point of view (b): the phase S satisfies an eikonal equation and the densities $|\mathbf{U}_k|^2$ satisfy transport equations.
- **Lagrangian** point of view (c): $(\mathbf{x}, \nabla_{\mathbf{x}} S)$ is given as the solution of the associated Hamiltonian system (ray tracing).



(a) Correct solution (b) Eikonal equation (c) Ray tracing

Ray methods 2/3

Eulerian point of view

- Plugging the WKBJ ansatz into the elastic wave equation:

$$\mathbf{H}(\mathbf{s}, \boldsymbol{\nabla}_{\mathbf{s}} S)\mathbf{U}_0 = \mathbf{0}\,, \qquad \text{[eikonal]}$$

$$\boldsymbol{\nabla}_{\mathbf{s}} \cdot \left(\mathbf{U}_0^{\mathsf{T}} \boldsymbol{\nabla}_{\boldsymbol{\xi}} \mathbf{H}(\mathbf{s}, \boldsymbol{\nabla}_{\mathbf{s}} S)\mathbf{U}_0\right) = 0\,, \qquad [0^{\mathrm{th}}\text{-order transport}]$$

where:

$$\begin{aligned} \mathbf{H}(\mathbf{s}, \boldsymbol{\xi}) &= \varrho(\mathbf{x})\left(\boldsymbol{\Gamma}(\mathbf{x}, \mathbf{k}) - \omega^2 \mathbb{I}_n\right) \\ &= \varrho(\mathbf{x}) \sum_{\alpha=1}^{n} (\lambda_\alpha^2(\mathbf{x}, \mathbf{k}) - \omega^2)\mathbf{b}_\alpha \otimes \mathbf{b}_\alpha \end{aligned}$$

is the dispersion matrix of the medium, $\mathbf{s} = (\mathbf{x}, t) \in \mathcal{O} \times \mathbb{R}$, $\boldsymbol{\xi} = (\mathbf{k}, \omega) \in \mathbb{R}^d \times \mathbb{R}$.

- Thus $\mathcal{H} = \det \mathbf{H} = 0 = \prod_{\alpha=1}^{n} \mathcal{H}_\alpha$ where $\mathcal{H}_\alpha = \varrho\left(\lambda_\alpha^2 - \omega^2\right)$, and therefore:

$$\boxed{\mathcal{H}_\alpha(\mathbf{s}, \boldsymbol{\nabla}_{\mathbf{s}} S) = 0}\,.$$

Ray methods 3/3

Lagrangian point of view

- Solving the eikonal equation for $S \in \mathbb{R}$ by the method of characteristics, $(\mathbf{s}, \boldsymbol{\nabla}_{\mathbf{s}} S)$ is given as a solution of the system:

$$\begin{aligned} \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}\tau} &= \boldsymbol{\nabla}_{\boldsymbol{\xi}} \mathcal{H}_\alpha(\mathbf{s}(\tau), \boldsymbol{\xi}(\tau)) \,, & \mathbf{s}(0) &= \mathbf{s}_0 \,, \\ \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}\tau} &= -\boldsymbol{\nabla}_{\mathbf{s}} \mathcal{H}_\alpha(\mathbf{s}(\tau), \boldsymbol{\xi}(\tau)) \,, & \boldsymbol{\xi}(0) &= \boldsymbol{\xi}_0 \neq \mathbf{0} \,, \end{aligned}$$

  in $T^*(\mathcal{O} \times \mathbb{R}) \setminus \{(\mathbf{s}, \boldsymbol{\xi});\, \boldsymbol{\xi} = \mathbf{0}\}$, with $\boldsymbol{\xi}_0 = \boldsymbol{\nabla}_{\mathbf{s}} S(\mathbf{s}_0)$.
- It follows immediately that $\mathcal{H}_\alpha(\mathbf{s}(\tau), \boldsymbol{\xi}(\tau)) = \mathcal{H}_\alpha(\mathbf{s}_0, \boldsymbol{\xi}_0) = \mathrm{C}^{\mathrm{st}}$ along the bicharacteristic curves $\tau \mapsto (\mathbf{s}(\tau), \boldsymbol{\xi}(\tau))$ solving the above system.
- The rays are defined as the projections on $\mathcal{O} \times \mathbb{R}_t$ of these curves, that is $\tau \mapsto \mathbf{s}(\tau)$.

Research program

- SEA, VCA: approximations (provably wrong);
- Ray methods: superposition principle, regularity... and they give only **one particular** construction of oscillating solutions.

Kinetic models: **all** oscillating solutions
The **key idea**: phase space description
(which also accounts for the directions in which waves propagate)

- Mechanical modeling for aerospace structures: material heterogeneity and damping, boundary conditions, higher-order kinematics;
- Numerical simulations;
- (Further modeling: diffusion limits).

Outline

Kinetic modeling

Reversible system

- Consider the bicharacteristic strip $\tau \mapsto (\mathbf{s}(\tau), \boldsymbol{\xi}(\tau))$ as the paths in phase space of some energy "particles" of which density is denoted by $\mathrm{W}(\mathbf{s}(\tau), \boldsymbol{\xi}(\tau))$, the latter shall satisfy:

$$\boxed{\frac{\mathrm{dW}}{\mathrm{d}\tau} = \{\mathrm{W}, \mathcal{H}\} = 0}\,, \qquad \text{[TE]}$$

  where $\{f, g\} = \boldsymbol{\nabla}_{\boldsymbol{\xi}} f \cdot \boldsymbol{\nabla}_{\mathbf{s}} g - \boldsymbol{\nabla}_{\mathbf{s}} f \cdot \boldsymbol{\nabla}_{\boldsymbol{\xi}} g$ (Poisson bracket).

- But $\frac{\mathrm{d}\omega}{\mathrm{d}\tau} = 0$ and $\mathbf{H} = \sum \mathcal{H}_\alpha \mathbf{b}_\alpha \otimes \mathbf{b}_\alpha$ with $\mathbb{I}_n = \sum \mathbf{b}_\alpha \otimes \mathbf{b}_\alpha$, thus:

$$\boxed{\mathrm{W} = \sum_{\alpha=1}^{n} \mathrm{w}_\alpha \delta(\mathcal{H}_\alpha)}\,.$$

- The link between W and the sequence $(\mathbf{u}_\varepsilon)$ is established by the Wigner measure of the latter.

Kinetic modeling

Irreversible system

- GENERIC framework:

$$\frac{\mathrm{dW}}{\mathrm{d}\tau} = \{\mathrm{W}, \mathcal{H}\} + [\mathrm{W}, \mathcal{S}] = 0\,, \qquad \text{[ITE]}$$

  where $\mathcal{S}$ is the entropy and $[\mathcal{S}, \mathrm{W}]$ is the dissipative bracket, satisfying the degeneracy conditions:

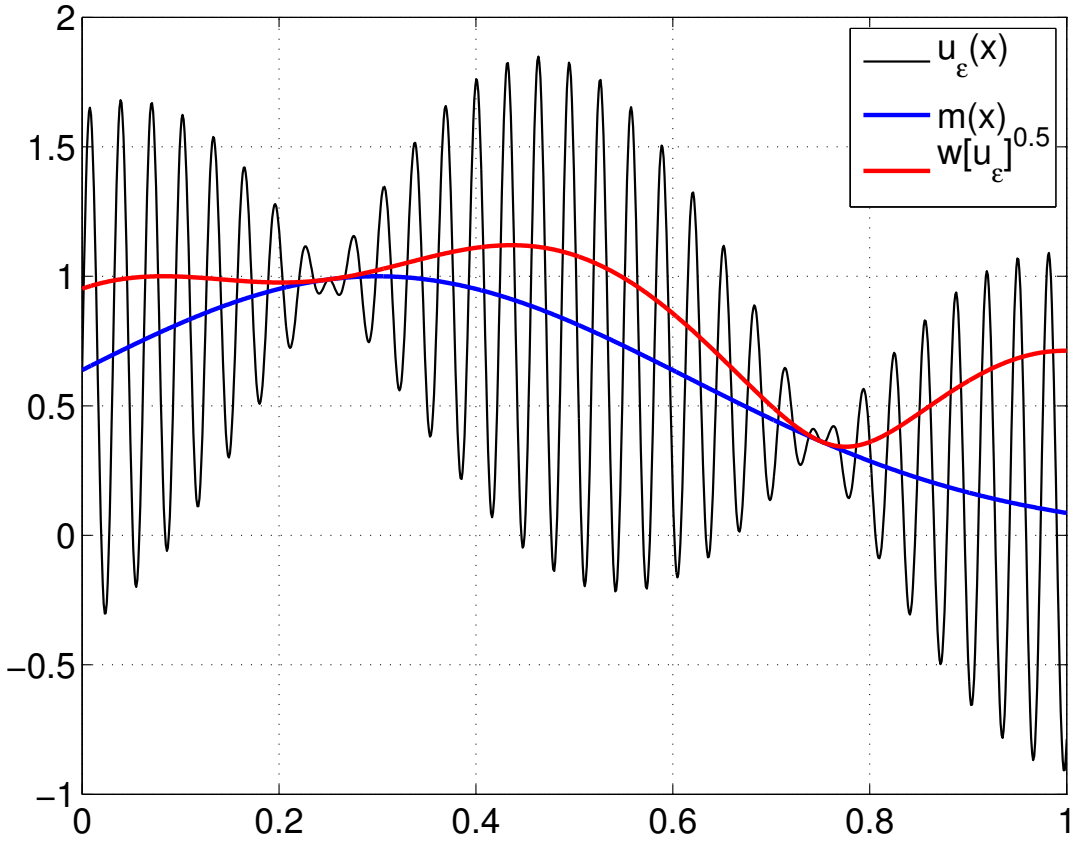
$$\{\mathcal{S}, \mathrm{W}\} = [\mathcal{H}, \mathrm{W}] = 0\,.$$

- The link between W and the sequence $(\mathbf{u}_\varepsilon)$ is established by the Wigner measure of the latter considering the wave equation with a time-dependent acoustic tensor.

Grmela-Öttinger PRE 56(6) 6620, 1997

Öttinger-Grmela PRE 56(6) 6633, 1997

Why quadratic observables?

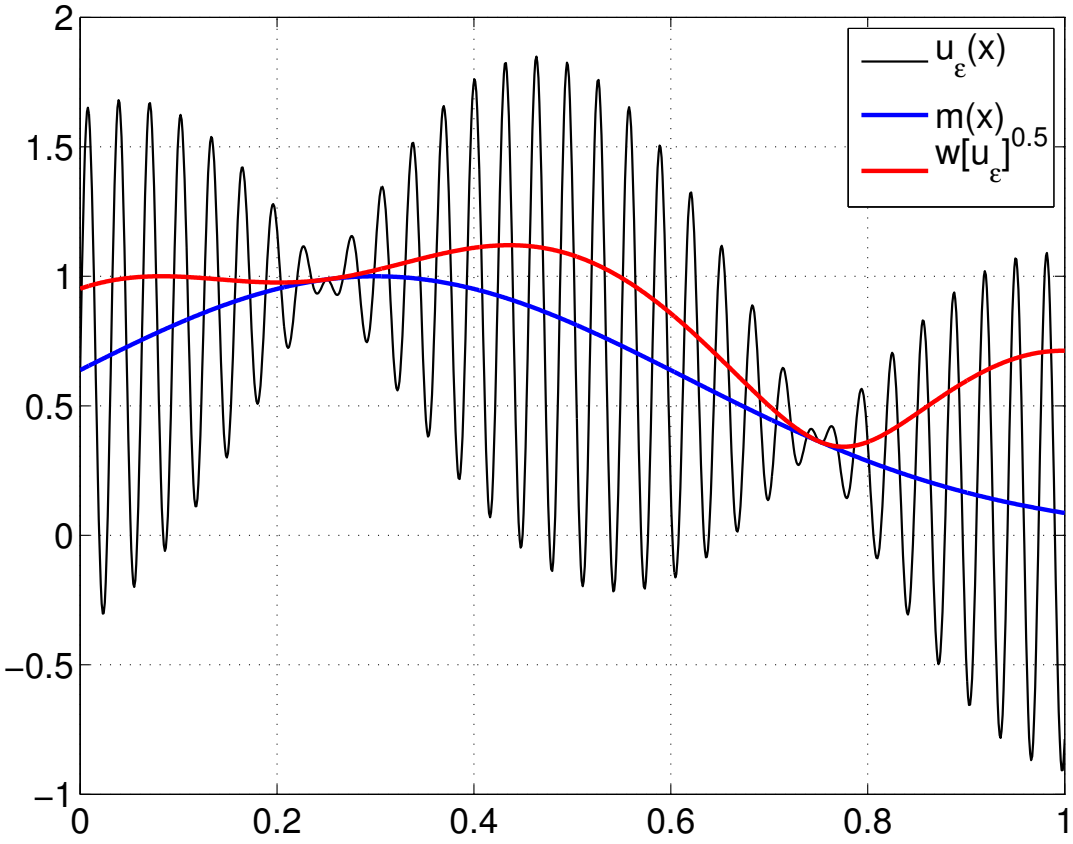


- Let:

$$u_\varepsilon(x) = m(x) + \sigma(x) \sin \frac{x}{\varepsilon} \,,$$

then $(u_\varepsilon) \rightharpoonup m$ weakly in $L^2(\mathbb{R})$ as $\varepsilon \to 0$, but (u_ε) has no strong limit in any L^p.

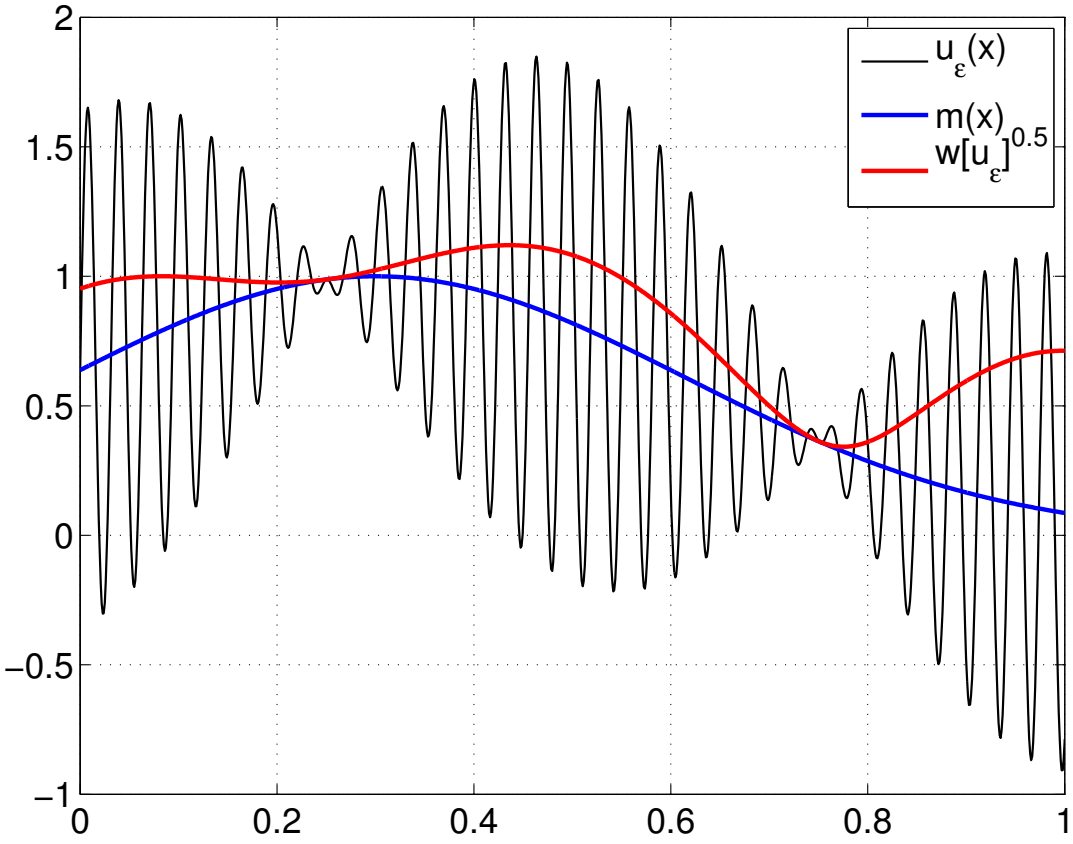
Why quadratic observables?



- Now for any observable $\varphi \in \mathcal{C}_0(\mathbb{R})$:

$$\lim_{\varepsilon \to 0} (\varphi(x) u_\varepsilon, u_\varepsilon)_{L^2} = \int_{\mathbb{R}} \varphi(x) \left((m(x))^2 + \frac{1}{2} (\sigma(x))^2 \right) \mathrm{d}x \,.$$

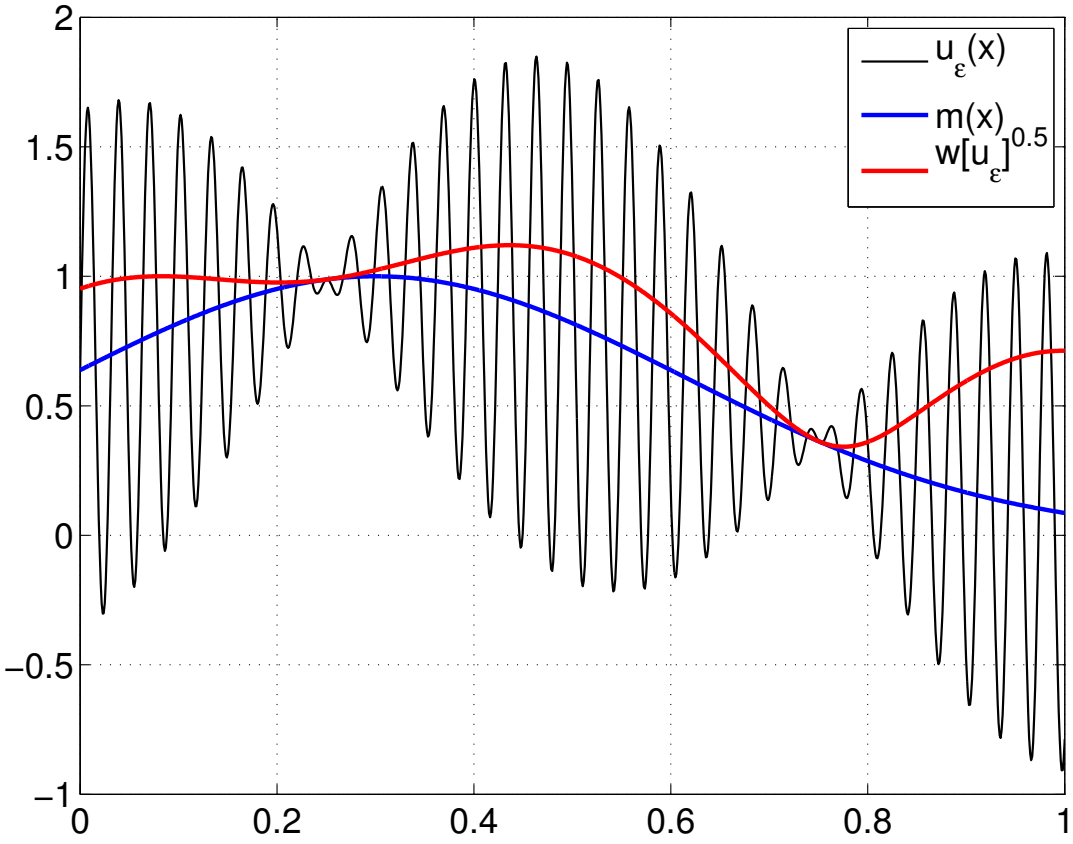
Why quadratic observables?



- Take an observable of the form:

$$\varphi(x, \partial_x) u_\varepsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}k\cdot x} \, \varphi(x, \mathrm{i}k) \widehat{u}_\varepsilon(k) \mathrm{d}k \, .$$

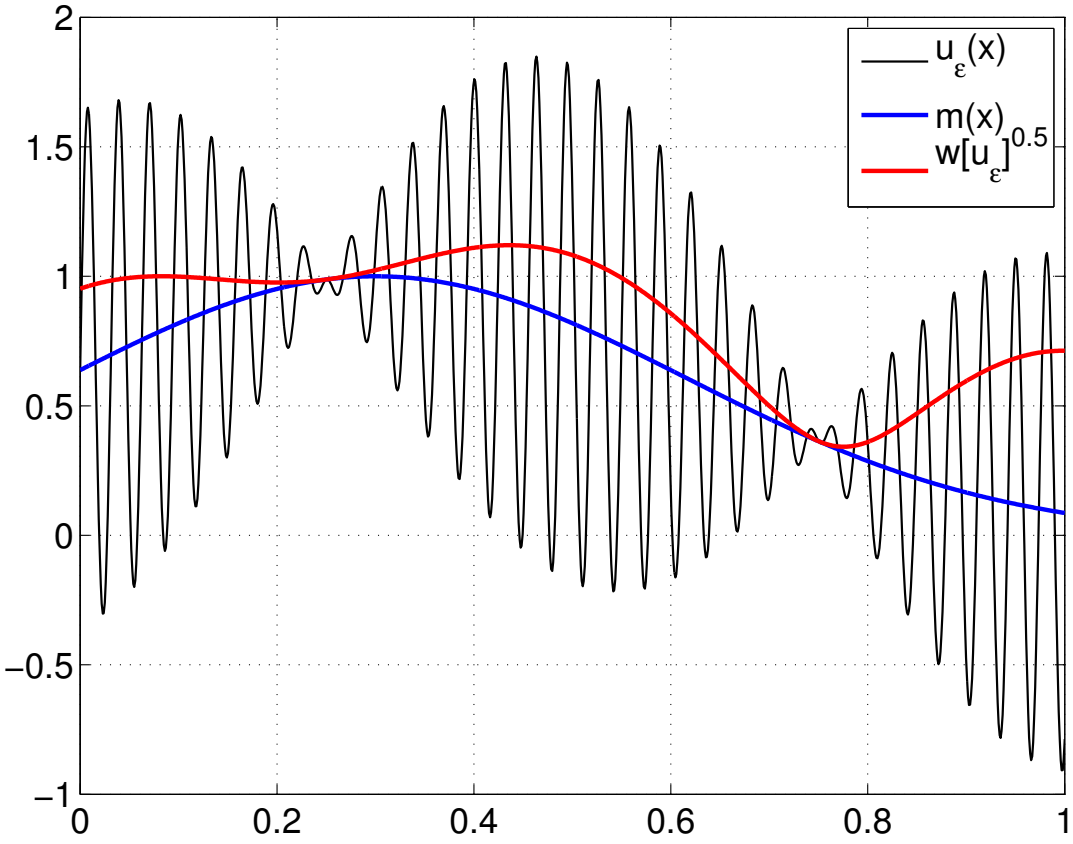
Why quadratic observables?



- Take an observable of the form:

$$\varphi(x, \varepsilon\partial_x)u_\varepsilon(u) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}k\cdot x}\,\varphi(x, \mathrm{i}\varepsilon k)\widehat{u}_\varepsilon(k)\mathrm{d}k\,.$$

Why quadratic observables?



- Then:

$$\lim_{\varepsilon \to 0} (\varphi(x, \varepsilon\partial_x) u_\varepsilon, u_\varepsilon)_{L^2} = \iint_{\mathbb{R}^2} \varphi(x, \mathrm{i}k) \mathrm{W}[u_\varepsilon](\mathrm{d}x, \mathrm{d}k)\,,$$

where $\mathrm{W}[u_\varepsilon]$ is the (positive) Wigner measure of (u_ε).

Wigner measure

Example

- **Example**: the overall kinetic and strain energy densities are in the high-frequency limit $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon(\mathbf{x},t) = \frac{1}{2}\varrho(\mathbf{x}) \int_{\mathbb{R}^{\boldsymbol{d}}} \operatorname{Tr} \mathbf{W}[\varepsilon\partial_{\boldsymbol{t}}\mathbf{u}_\varepsilon(\cdot,t)](\mathbf{x},\mathrm{d}\mathbf{k}) \,,$$

$$\lim_{\varepsilon\to 0} \mathcal{U}_\varepsilon(\mathbf{x},t) = \frac{1}{2}\varrho(\mathbf{x}) \int_{\mathbb{R}^{\boldsymbol{d}}} \boldsymbol{\Gamma}(\mathbf{x},\mathbf{k}) : \mathbf{W}[\mathbf{u}_\varepsilon(\cdot,t)](\mathbf{x},\mathrm{d}\mathbf{k}) \,,$$

(eventually one can prove that they are equal in this very limit), or

$$\boxed{\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\mathbf{x},t) = \sum_{\alpha=1}^{\boldsymbol{n}} \int_{\mathbb{R}^{\boldsymbol{d}}} \mathrm{w}_\alpha(\mathbf{x},t;\mathrm{d}\mathbf{k})} \,.$$

Boundary conditions

Rankine-Hugoniot condition

- Let a discontinuity front Σ_D be defined by the hypersurface in phase space:

$$\Sigma_D = \{(\mathbf{s}, \boldsymbol{\xi}) \in T^*(\mathcal{O} \times \mathbb{R});\ \Sigma(\mathbf{s}, \boldsymbol{\xi}) = 0\}\,.$$

- The Rankine-Hugoniot condition associated to piecewise continuous (weak) solutions of the Liouville equation [TE] on $\Sigma_D \cap \{\mathcal{H}(\mathbf{s}, \boldsymbol{\xi}) = 0\}$:

$$\boxed{[\![\{\mathcal{H}, \Sigma\}\mathrm{W}]\!] = 0}\,.$$

- **Consequence**: continuity of the overall normal power flow across any fixed interface $\{\Sigma(\mathbf{x}, \mathbf{k}) = 0\}$,

$$\sum_{\alpha=1}^{n} [\![\{\lambda_\alpha, \Sigma\}\mathrm{w}_\alpha]\!] = 0\,.$$

Boundary conditions

Snell-Descartes law

- The Hamiltonian $\mathcal{H} = 0$ across Σ_D at t_0 such that $\Sigma(\mathbf{s}(t_0), \boldsymbol{\xi}(t_0)) = 0$:

$$\lambda_\alpha(\mathbf{x}(t_0^-), \mathbf{k}(t_0^-)) = \lambda_\beta(\mathbf{x}(t_0^+), \mathbf{k}(t_0^+)), \quad 1 \leq \alpha, \beta \leq n\,.$$

- **Consequence**: Snell-Descartes law of diffraction,

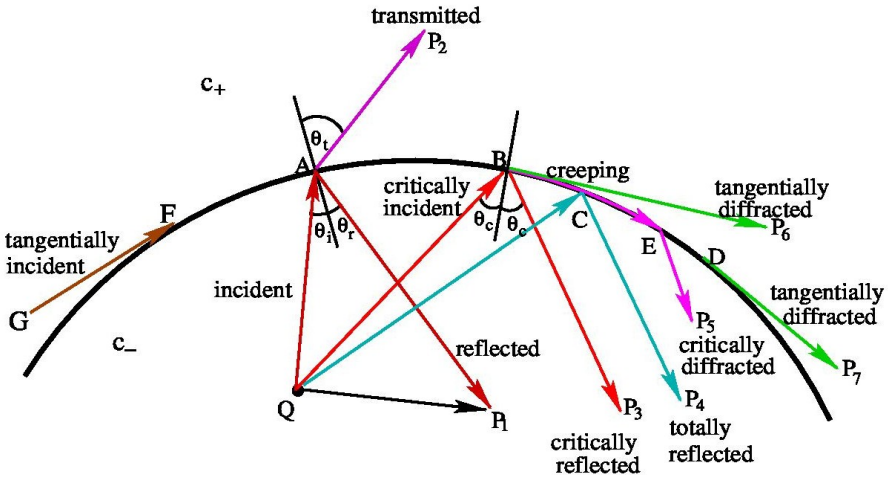
$$\boxed{P_{\mathbf{x}}\mathbf{k} = \text{constant}}\,,$$

where $P_{\mathbf{x}} = \mathbb{I}_d - \hat{\mathbf{n}}(\mathbf{x}) \otimes \hat{\mathbf{n}}(\mathbf{x})$ is the orthogonal projection on the tangent plane to a fixed interface at $\mathbf{x}$.

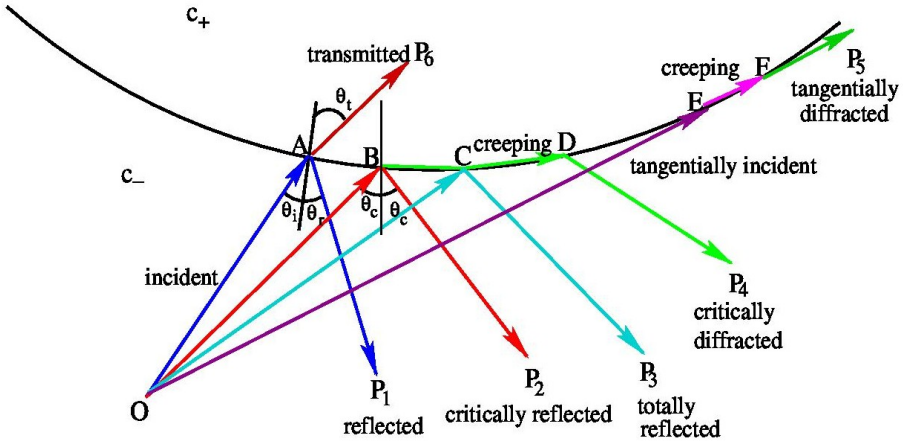
Boundary conditions

Reflection/transmission coefficients

- **Example**: acoustic waves $\lambda_1(\mathbf{x}, \mathbf{k}) = \pm c(\mathbf{x})|\mathbf{k}|$ $(n = 1)$.



Convex acoustic interface with $c_+ > c_-$



Concave acoustic interface with $c_+ > c_-$

Jin-Yin JCP 227(12) 6106, 2008

Boundary conditions

Reflection/transmission coefficients

- $\{\mathcal{H}, \Sigma\} > 0$: transverse reflections (the hyperbolic set);
- $\{\mathcal{H}, \Sigma\} < 0$: total reflections (the elliptic set);
- $\{\mathcal{H}, \Sigma\} = 0$: critical reflections (the glancing set):
  - $\{\mathcal{H}, \{\mathcal{H}, \Sigma\}\} > 0$: diffractive rays,
  - $\{\mathcal{H}, \{\mathcal{H}, \Sigma\}\} < 0$: gliding rays,
  - $\{\mathcal{H}, \{\mathcal{H}, \Sigma\}\} = 0$: higher-order gliding rays (w. inflection point).

- For elastic waves the situation is a mix of all cases!
- Boundary conditions raise profound mathematical issues, mostly unsolved to date. Unfortunately they have also great engineering relevance!

Miller JMPA 79(3) 227, 2000

Savin IMMIJ 1(1) 53, 2007

Le Guennec-Savin JASA 130(6) 3706, 2011

Akian-Alexandre-Bougacha KRM 4(3) 589, 2011

Akian ASY 78(1–2) 37, 2012

Outline

Discontinuous finite elements for the TE 1/2

- TE is a linear conservation equation in phase space $\Omega = \mathcal{O} \times \mathbb{R}^d \ni \mathbf{z} = (\mathbf{x}, \mathbf{k})$:

$$\mathcal{L}\mathrm{w}_\alpha \equiv \partial_t \mathrm{w}_\alpha + \boldsymbol{\nabla}_\mathbf{z} \cdot \mathcal{F}(\mathrm{w}_\alpha) = 0\,, \quad \mathcal{F}(\mathrm{w}_\alpha) = \begin{pmatrix} \boldsymbol{\nabla}_\mathbf{k}\lambda_\alpha \\ -\boldsymbol{\nabla}_\mathbf{x}\lambda_\alpha \end{pmatrix} \mathrm{w}_\alpha\,;$$

- Initial conditions: $\mathrm{w}_\alpha(\mathrm{d}\mathbf{z}; t = 0) = \mathrm{w}^0_\alpha(\mathrm{d}\mathbf{z})$;
- The modes α get coupled by the reflection/transmission conditions.

Discontinuous finite elements for the TE 2/2

- Consider the partition $\mathcal{T}_h = \cup_{r=1}^{\mathcal{N}} \Omega_r$ of Ω into $\mathcal{N}$ non-overlapping subdomains, or finite elements, and the trial space $\mathcal{W}_h$ of piecewise continuous functions on $\mathcal{T}_h$.
- Variational formulation using the test space $\mathcal{V}_h$:

1. "Weak" form: Find $\mathrm{w}_\alpha \in \mathcal{W}_h$ s.t.

$$\int_{\Omega_r} (\mathrm{v}\partial_t \mathrm{w}_\alpha - \boldsymbol{\nabla}_{\mathbf{z}} \mathrm{v} \cdot \mathcal{F}(\mathrm{w}_\alpha))\, \mathrm{d}\Omega + \int_{\partial\Omega_r} \mathrm{v}\mathcal{F}_r^*(\mathrm{w}_\alpha^-, \mathrm{w}_\beta^+) \cdot \hat{\mathbf{n}}_r \, \mathrm{d}\gamma(\mathbf{z}) = 0\,, \quad \forall \mathrm{v} \in \mathcal{V}_h\,,$$

where $\mathcal{F}(\mathrm{w}_\alpha) \cdot \hat{\mathbf{n}}|_{\partial\Omega_r}$ is replaced by $\mathcal{F}_r^*(\mathrm{w}_\alpha^-, \mathrm{w}_\beta^+) \cdot \hat{\mathbf{n}}_r$ because w_α may have two different values on both sides of the edge $\partial\Omega_r$.

Discontinuous finite elements for the TE 2/2

- Consider the partition $\mathcal{T}_h = \cup_{r=1}^{\mathcal{N}} \Omega_r$ of Ω into $\mathcal{N}$ non-overlapping subdomains, or finite elements, and the trial space $\mathcal{W}_h$ of piecewise continuous functions on $\mathcal{T}_h$.
- Variational formulation using the test space $\mathcal{V}_h$:

2. "Strong" form (Lesaint & Raviart 1974): Find $\mathrm{w}_\alpha \in \mathcal{W}_h$ s.t.

$$\int_{\Omega_r} \mathrm{v}\mathcal{L}\mathrm{w}_\alpha \, \mathrm{d}\Omega = \int_{\partial\Omega_r} \mathrm{v} \left(\mathcal{F}(\mathrm{w}_\alpha^-) - \mathcal{F}_r^*(\mathrm{w}_\alpha^-, \mathrm{w}_\beta^+) \right) \cdot \hat{\mathbf{n}}_r \, \mathrm{d}\gamma(\mathbf{z}) \,, \quad \forall \mathrm{v} \in \mathcal{V}_h \,,$$

e.g. penalization methods.

Discontinuous finite elements for the TE 2/2

- Consider the partition $\mathcal{T}_h = \cup_{r=1}^{\mathcal{N}} \Omega_r$ of Ω into $\mathcal{N}$ non-overlapping subdomains, or finite elements, and the trial space $\mathcal{W}_h$ of piecewise continuous functions on $\mathcal{T}_h$.
- Variational formulation using the test space $\mathcal{V}_h$:

3. "Ultra-weak variational formulation":

$$\partial_t \left(\int_{\Omega_r} \mathrm{v}\mathrm{w}_\alpha \, \mathrm{d}\Omega \right) + \int_{\partial\Omega_r} \mathrm{v}\mathcal{F}_r^*(\mathrm{w}_\alpha^-, \mathrm{w}_\beta^+) \cdot \hat{\mathbf{n}}_r \, \mathrm{d}\gamma(\mathbf{z}) = 0 \,, \quad \forall \mathrm{v} \in \mathcal{V}_h \,,$$

choosing $\mathcal{V}_h = \{\mathrm{v};\ \mathcal{L}^*\mathrm{v} = 0 \text{ in } \Omega_r\}$ (plane waves for example are $\mathrm{v}(\mathbf{z}, t) = \mathbf{V}\phi(\mathbf{k}\cdot\mathbf{x} - \omega t)$ such that $\mathbf{\Gamma}(\mathbf{z})\mathbf{V} = \omega\mathbf{V}$, with the acoustic tensor $\mathbf{\Gamma}(\mathbf{z}) := \mathrm{diag}\{\lambda_\alpha(\mathbf{z})\}_{1\leq\alpha\leq n}$).

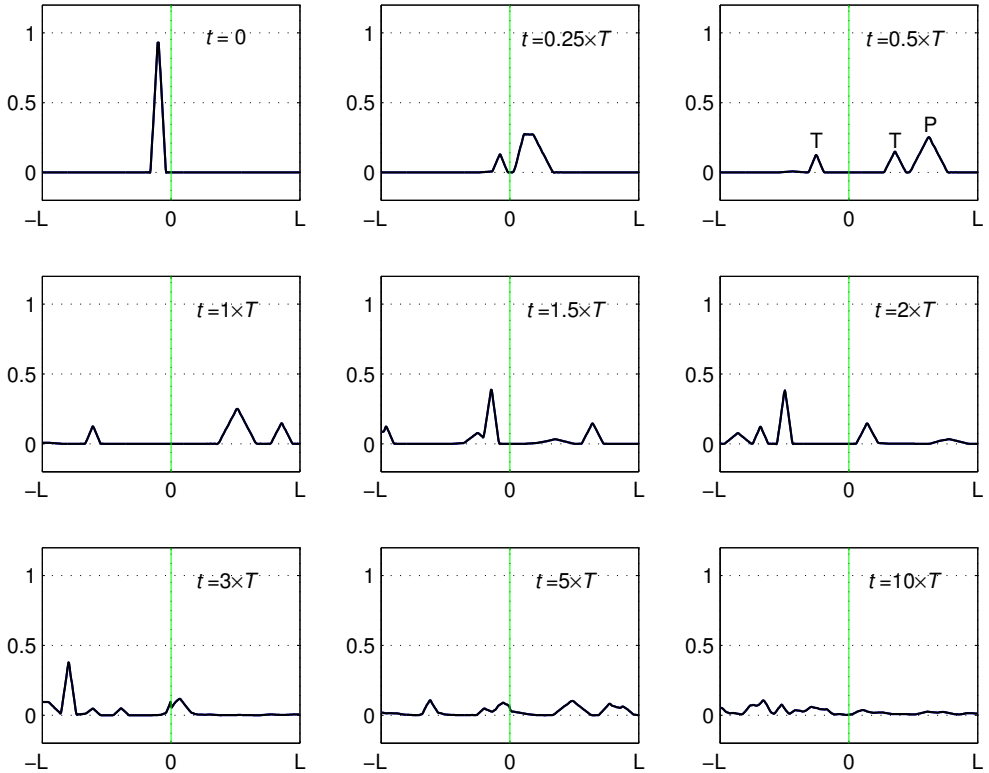
Després CRAS I 318 939, 1994

References

Further reading...

- Cockburn-Karniadakis-Shu (eds.): *Discontinuous Galerkin Methods: Theory, Computation and Applications*, Springer (2000)
- Karniadakis-Sherwin: *Spectral/hp Element Methods For Computational Fluid Dynamics* (2nd ed.), Springer (2005)
- Li: *Discontinuous Finite Elements in Fluid Dynamics and Heat Transfer*, Springer (2006)
- Hesthaven-Warburton: *Nodal Discontinuous Galerkin Methods*, Springer (2008)
- Rivière: *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations*, SIAM (2008)
- Di Pietro-Ern: *Mathematical Aspects of Discontinuous Galerkin Methods*, Springer (2012)

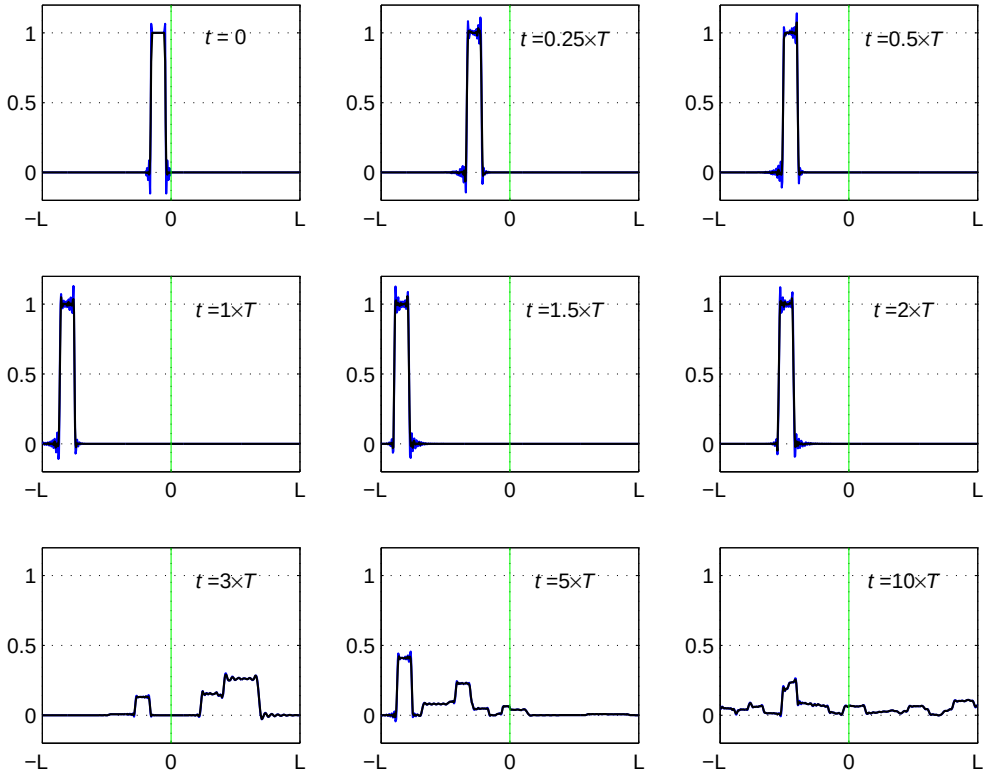
Example #1: beam junction



Runge-Kutta discontinuous FEM
w. Legendre modal expansion
$\mathcal{N} = 40$, $N = 10$, RK-SSP$(8, 8)$
— unfiltered
— 1st-order exponential filter

Energy transport in a beam junction with $\phi = 60°$, $\frac{E_2}{E_1} = 2$, $\nu_1 = \nu_2 = 0.3$, *and* $T = \frac{L}{c_{T2}}$.

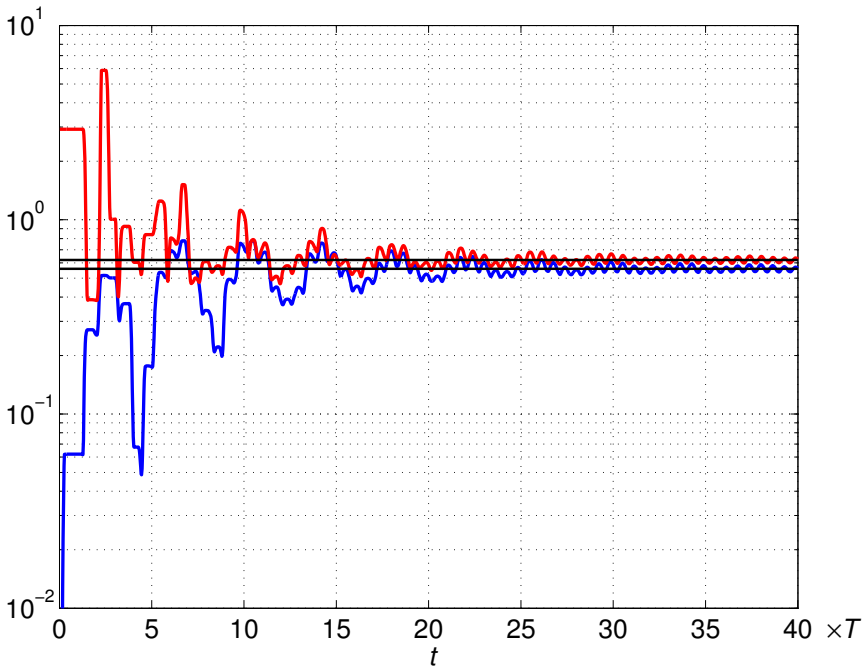
Example #1: beam junction



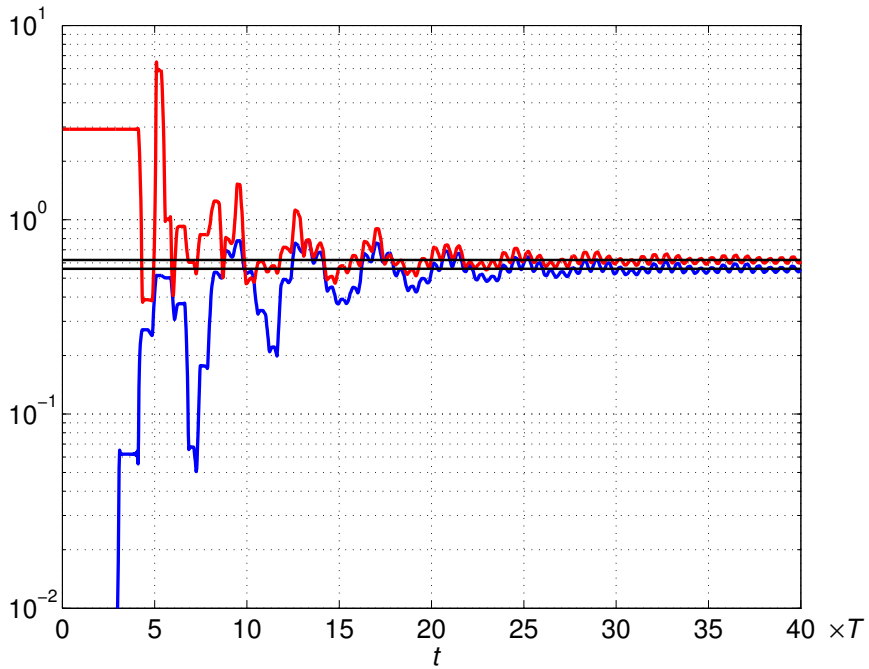
Runge-Kutta discontinuous FEM
w. Gauss-Lobatto nodal expansion
$\mathcal{N} = 40$, $N = 10$, RK-SSP$(8, 8)$
— unfiltered
— 0th-order exponential filter

Energy transport in a beam junction with $\phi = 60°$, $\frac{E_2}{E_1} = 2$, $\nu_1 = \nu_2 = 0.3$, *and* $T = \frac{L}{c_{T2}}$.

Example #1: beam junction



"Hat" source: Legendre modal expansion
$\mathcal{N} = 40,\ N = 5,$ RK-SSP$(5,4)$

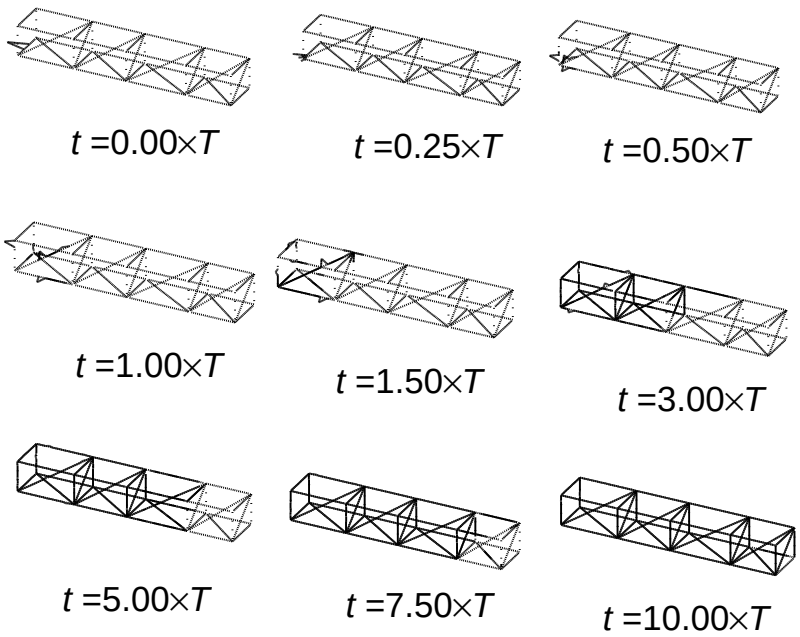


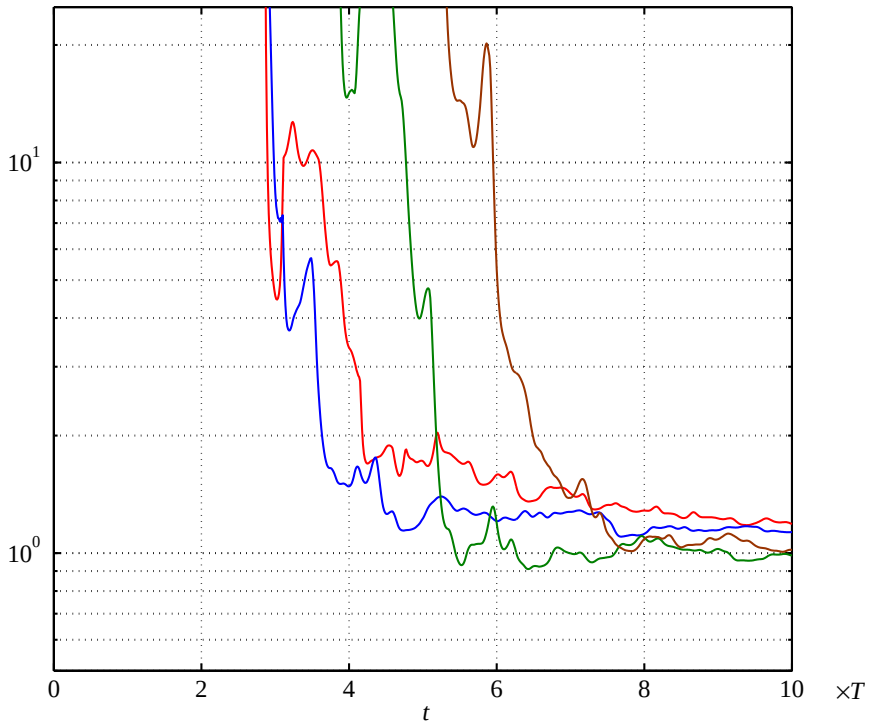
"Square" source: nodal expansion
$\mathcal{N} = 40,\ N = 5,$ RK-SSP$(5,4)$

Energy ratios $t \mapsto \frac{\mathcal{E}_{\mathrm{P}}(t)}{\mathcal{E}_{\mathrm{T}}(t)}$ *within each sub-structure (— beam #1, — beam #2);* $\phi = 60°$, $\frac{E_2}{E_1} = 2,\ \nu_1 = 0.4,\ \nu_2 = 0.1,\ T = \frac{L}{c_{\mathrm{T}2}}$.

Savin PEM 28 194, 2012

Example #2: beam truss





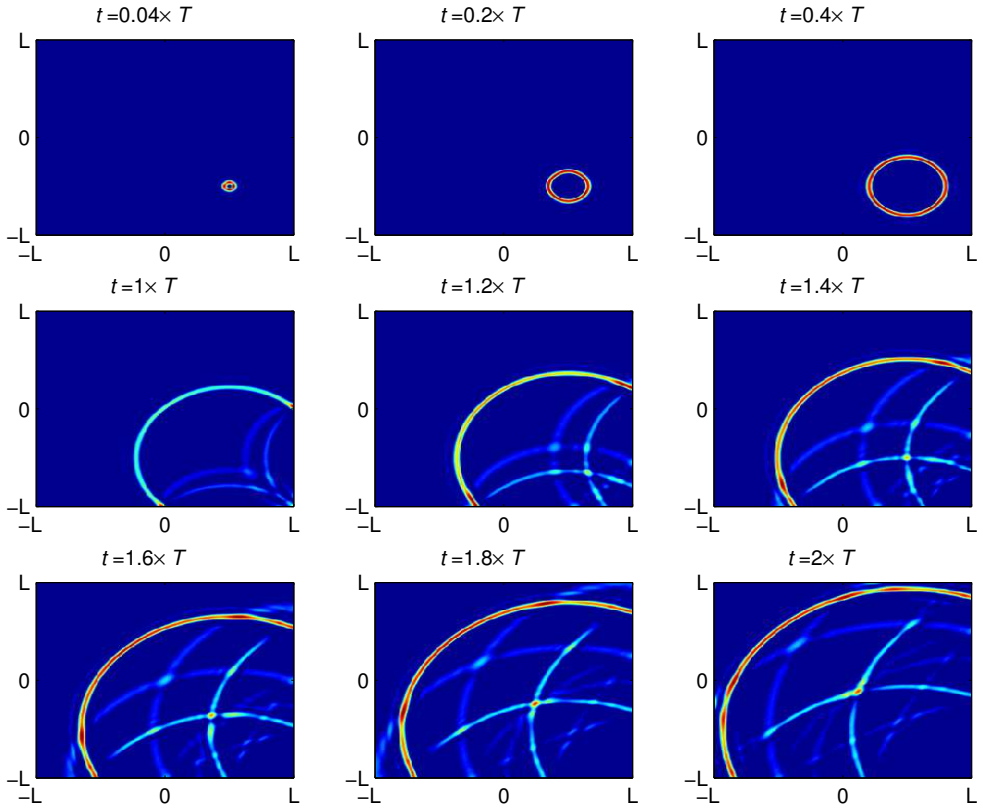
Runge-Kutta discontinuous FEM
w. Legendre modal expansion
$\mathcal{N} = 1212$, $N = 8$, RK-SSP(8,8) ($\mathrm{CFL} = 10^{-2}$)

$$\frac{\mathcal{E}_{\mathrm{T}}^{r}(t)}{\mathcal{E}_{\mathrm{P}}^{r}(t)} \underset{t\to+\infty}{\longrightarrow} \frac{2c_{\mathrm{P}}}{c_{\mathrm{T}}} \quad (1 \leq r \leq 4)$$

Energy transport in a beam truss with $T = \frac{L}{c_{\mathrm{T}}}$.

Le Guennec-Savin JASA 130(6) 3706, 2011

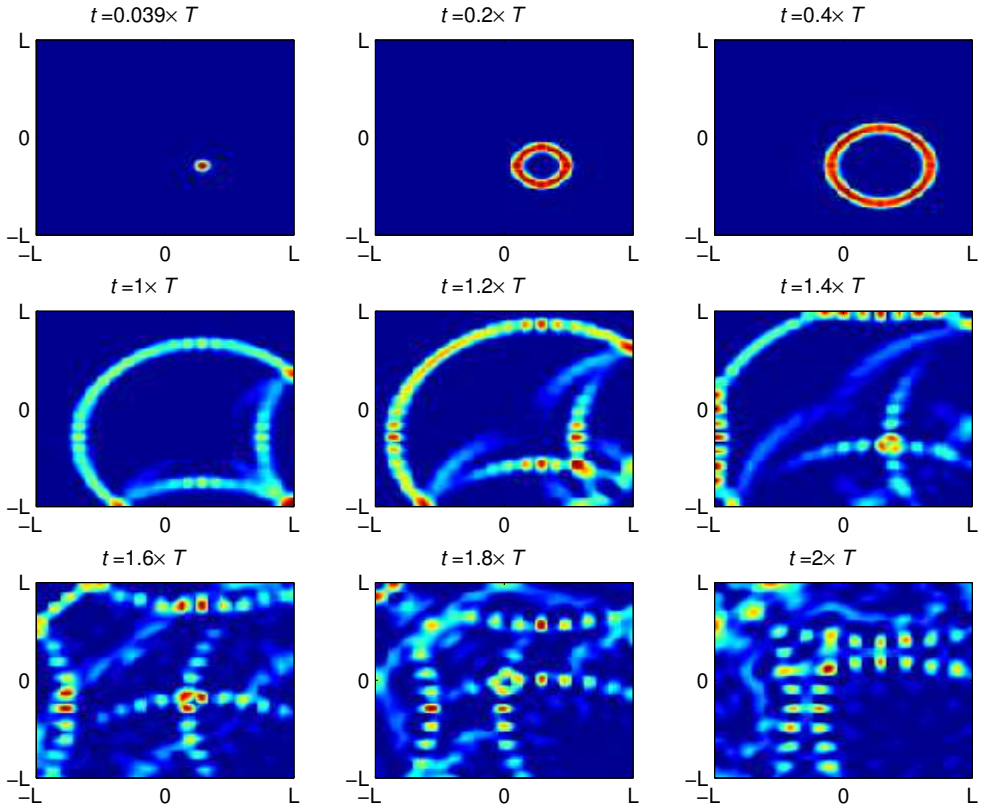
Example #3: thick plate



Runge-Kutta discontinuous FEM
w. mixed nodal/modal expansion
$\mathcal{N} = 21 \times 21$, $N = 5$ (LGL nodes), $P = 2$ (Fourier), SSP$(3,3)$

Energy transport in a thick plate with $\nu = 0.3$ and $T = \frac{L}{c_{\mathrm{S}}}$.
Issues with Fourier expansion: dispersion, positivity.

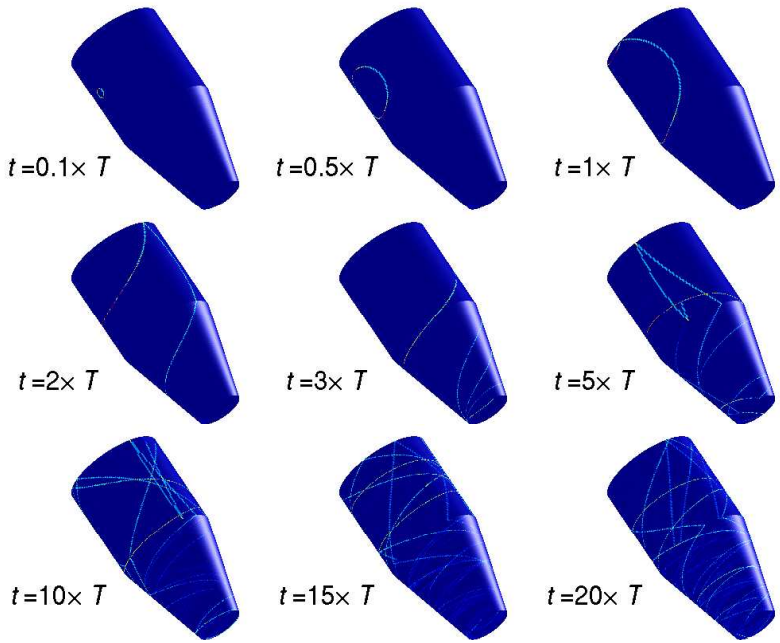
Example #3: thick plate

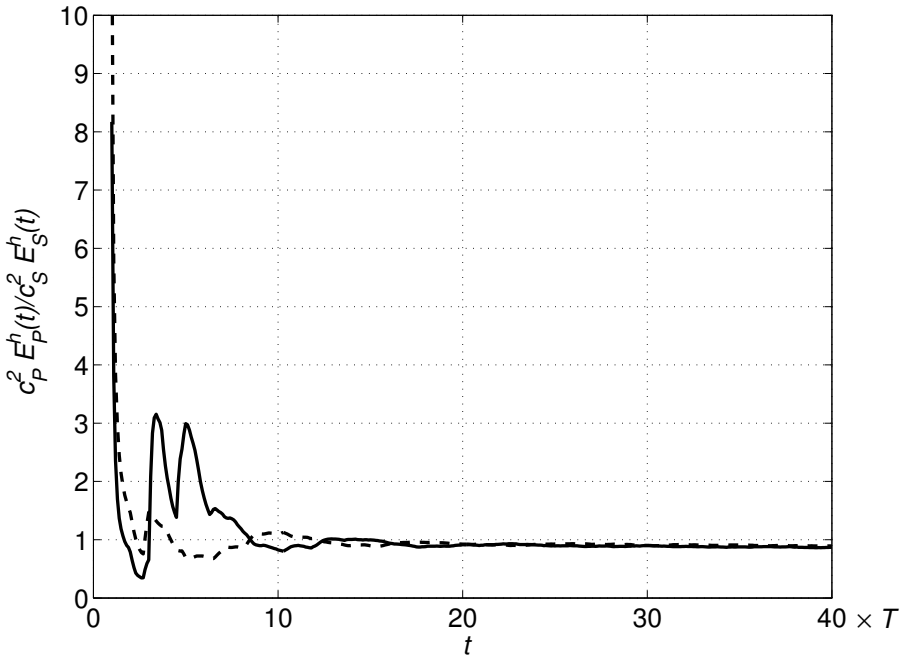


Runge-Kutta discontinuous FEM
w. full nodal expansion
$\mathcal{N} = 21 \times 21$, $N = 3$ (LGL nodes), $P = 59$ (Lagrange), SSP$(3, 3)$

Energy transport in a thick plate with $\nu = 0.3$ *and* $T = \frac{L}{c_{\mathrm{S}}}$.
Issue with Lagrange expansion: ray effect.

Example #4: transport in a shell junction





Direct Monte-Carlo method
10^6 sample paths

$$\frac{\mathcal{E}_{\mathrm{P}}^{r}(t)}{\mathcal{E}_{\mathrm{S}}^{r}(t)} \underset{t\to+\infty}{\longrightarrow} \frac{c_{\mathrm{S}}^2}{c_{\mathrm{P}}^2}$$ (— *shell #1*, -- *shell #2*)

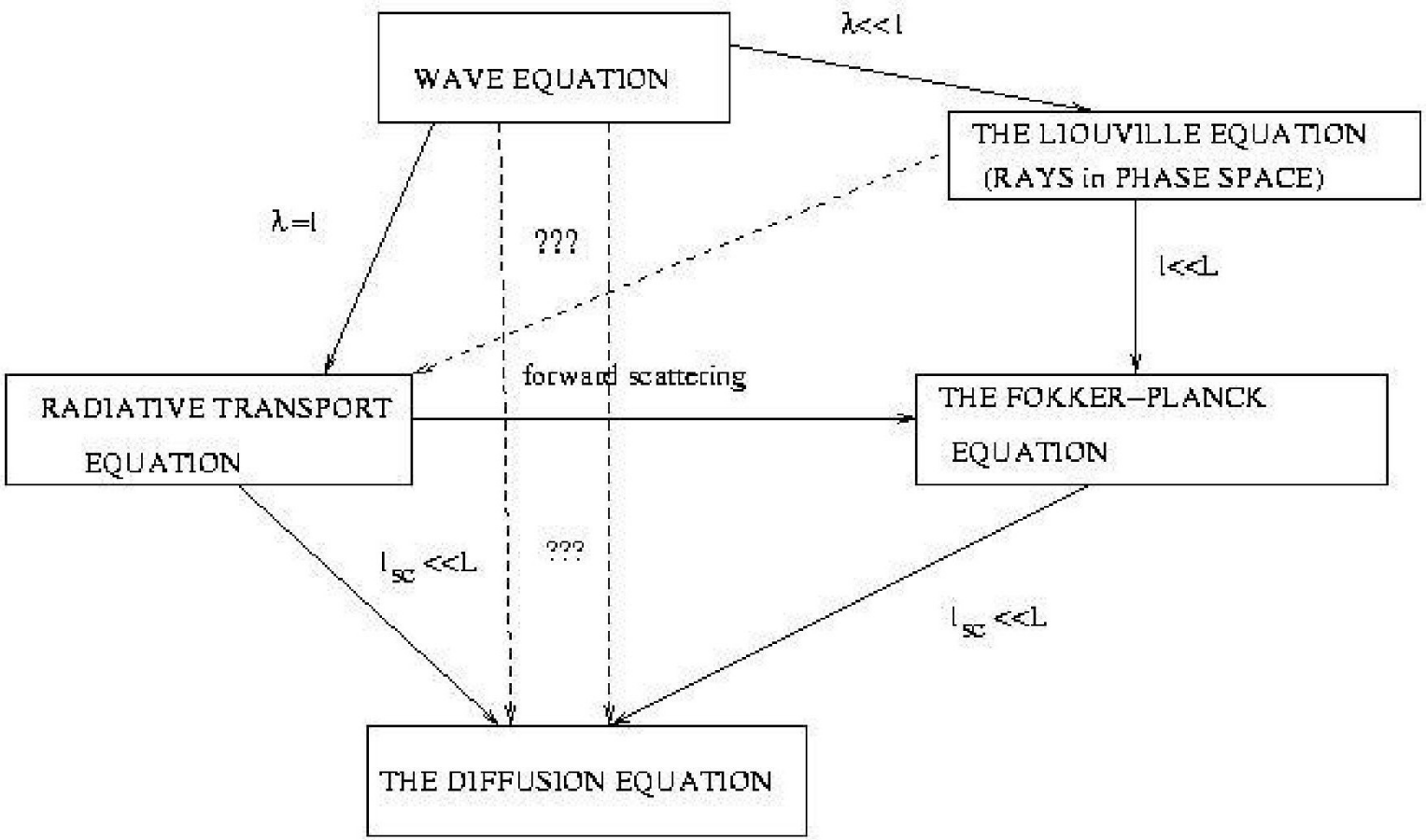
Energy transport in a shell junction impacted by a shear load with $\phi = 15°$, $\frac{E_2}{E_1} = 2$, $\nu_1 = 0.3$, $\nu_2 = 0.2$, $T = \frac{L}{c_{\mathrm{T}2}}$.

Savin PEM 28 194, 2012

Outline

Wave propagation in heterogeneous media

- **Scales**: λ – wavelength, ℓ – correlation length, L – propagation/observation distance...

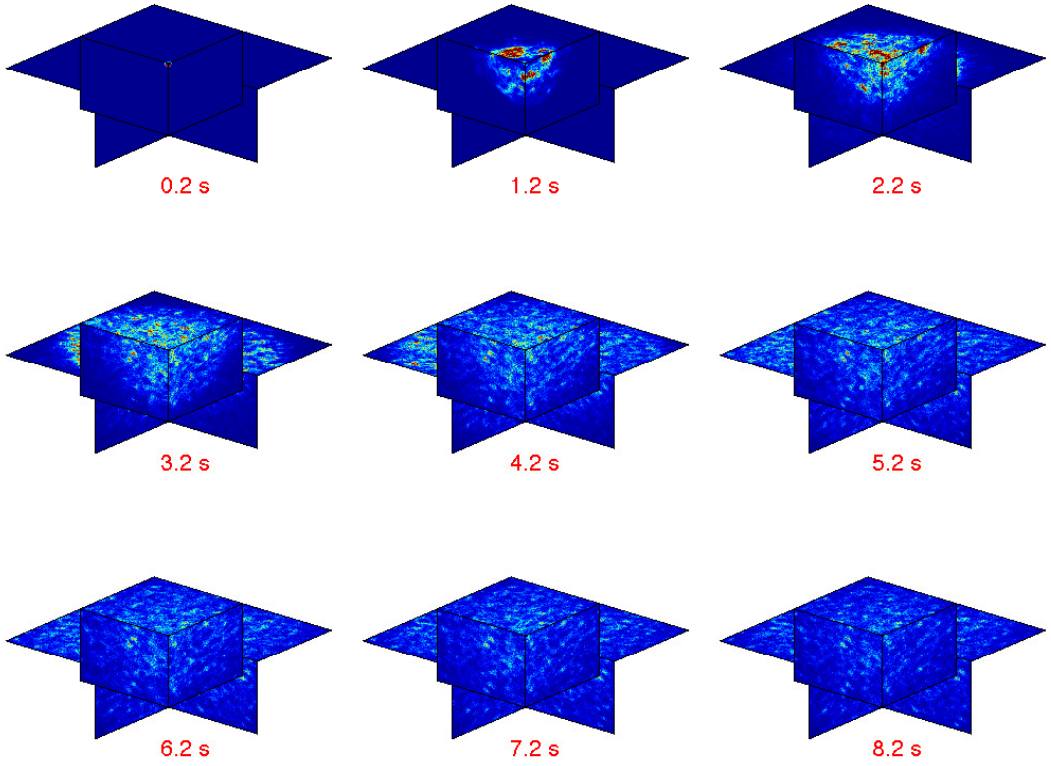


... and the **mesoscopic scale** $\ell_{\rm sc}$ – scattering mean free path.

Ryzhik HFWP'05, 2005

Radiative transfer in anisotropic media

Example: triclinic random medium



Spectral FEM LGL + PML
$7 \cdot 10^7$ ddls, $\mathcal{O} = 5 \times 5 \times 2.5$ km^3

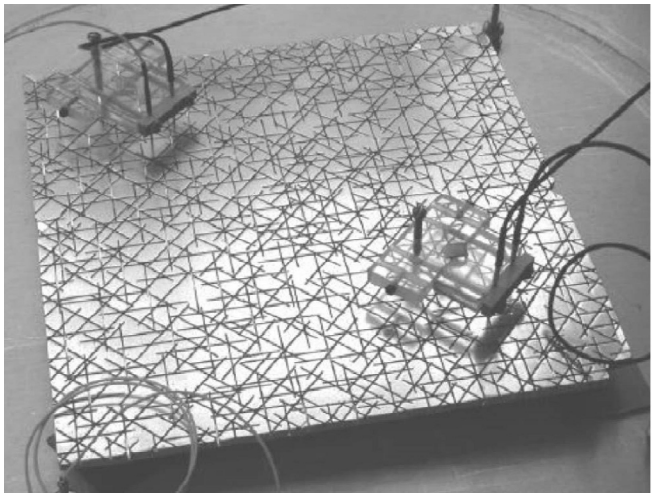
Velocity field in an anisotropic heterogeneous half-space with mean homogeneous isotropy, monopole on the surface ; $\underline{c}_{\mathrm{P}} = 2000$ m/s, $\underline{c}_{\mathrm{S}} = 1000$ m/s, $\underline{\varrho} = 2000$ kg/m^3.

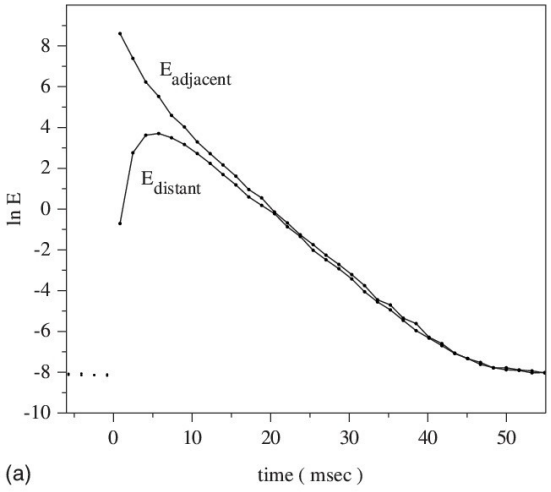
Ta-Clouteau-Cottereau EJCM 19(1-2-3) 241, 2010

Baydoun-Savin-Clouteau-Cottereau-Guilleminot, *in progress*

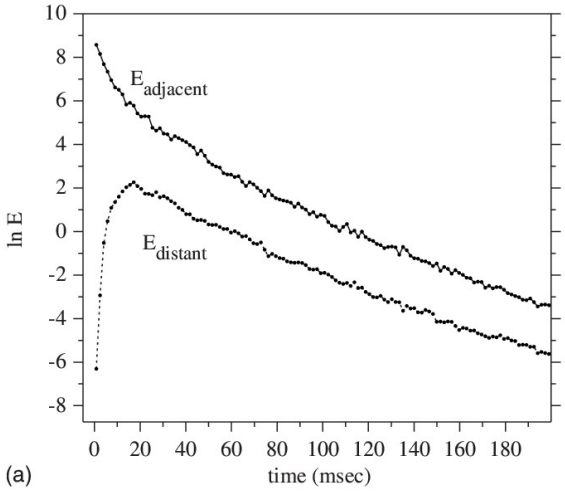
Radiative transfer in anisotropic media

Example: anisotropic diffusion





$f = 664 \pm 40$ kHz



$f = 195 \pm 40$ kHz

Privileged directions for diffusion corresponding to the principal axes of diffusion
$\rightarrow$ *localisation when* $\lambda \sim \ell_{\rm sc}$?

Lobkis-Weaver JASA 124(6) 3528, 2008

Perspectives

Modeling

- Diffraction and glancing;
- Diffusive regimes for slender structures $\rightarrow$ SEA;
- "Multi-physics" coupling with quadratic quantities.

Numerical simulations

- Regularized interface conditions by diffusion (Dafermos 1977, Fornet 2007);
- Discontinuous finite elements: phase-space discretization, positivity, filtering, goal-oriented adaptive strategies...
- "Multi-scale" coupling of transport & diffusion.

Hot topics

- RTN HYKE 2002–2005, GdR CHANT, GdR MésoIm@ge, GdR Ondes, GdR Ultrasons, ERC StG NuSiKiMo, ANR MNEC...
- CIRM'05, HFWP'05, Waves'07, Waves'09, HYKE'10, Waves'11, HYP'12, Euromech 540...
- AA, CiCP, CiMP, CiMS, JASA, JCP, JHDE, JMP, JSC, KRM, WM, WRCM...