Kinetic models of elastic waves in bounded media
with applications in structural acoustics

Éric Savin
eric.savin@{onera,ecp}.fr

ONERA – Simulation Numérique des Écoulements et Aéroacoustique
29, avenue de la Division Leclerc
F-92322 Châtillon cedex

ECP – Département Mécanique et Génie Civil
Grande Voie des Vignes
F-92295 Châtenay-Malabry
Outline

1. High-frequency vibrations
2. Ray methods
3. Elastic energy transport model
4. Numerical examples for slender structures
5. Perspectives
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1. High-frequency vibrations
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Frequency response function of complex structures

Example

- Cessna Citation fuselage: 9 floor/19 ceiling ribs, 22 stringers.
- Length = 2.55 m, radius = 0.81 m, thickness = 0.8-1.2 mm.

Herdic et al. JASA 117(6) 3667, 2005
Frequency response function of complex structures

Example

Phase distributions of surface velocity

Magnitude distributions of surface velocity

Excitation: point force at a rib/stringer stiffener intersection.
Structural-acoustics system

- **Hypotheses:** linear, visco-elastic materials with density $\rho(x)$ and relaxation tensor $C(x, t)$, $x \in \mathcal{O} \subseteq \mathbb{R}^d$, $t \in \mathbb{R}^+$.  

- **Local balance of momentum** for the displacement field $u$ in $\mathcal{O} \times \mathbb{R}$:

\[ \rho \partial^2_t u = \text{Div} \sigma. \]

- **Constitutive equation** for the stress field $\sigma$ in $\overline{\mathcal{O}} \times \mathbb{R}^+$ as a function of the strain field $\epsilon(u) = \nabla_x \otimes_s u$:

\[ \sigma(x, t) = C(x, 0)\epsilon(u) + \partial_t C(x, \cdot) *_t \epsilon(u) . \]

ineffective in the high-frequency limit.
“High-frequency” setting 1/2

- **High frequencies** correspond to $\varepsilon \to 0$ for strongly “$\varepsilon$–oscillatory” initial conditions:

  $u_\varepsilon(x, 0) = u_\varepsilon^0(x), \quad \|\varepsilon \nabla_x u_\varepsilon^0\|_{L^2_{\text{loc}}} < \infty,$

  and:

  $\partial_t u_\varepsilon(x, 0) = v_\varepsilon^0(x), \quad \|\varepsilon \nabla_x v_\varepsilon^0\|_{L^2_{\text{loc}}} < \infty.$

- **Example**: plane waves, $\varepsilon \equiv (|k|L)^{-1},$

  $u_\varepsilon^0(x) = \varepsilon A(x) e^{\frac{i}{\varepsilon} k \cdot x}, \quad v_\varepsilon^0(x) = B(x) e^{\frac{i}{\varepsilon} k \cdot x}.$
Example: impulse loads in spatial structures

- Example #1: pyrotechnic cut.

*Impulse loads* $\rightarrow$ *High frequency (HF) wave propagation.*
Example: impulse loads in spatial structures

- Example #2: solar panel unfolding.

\[ \text{Impulse loads} \rightarrow \text{High frequency (HF) wave propagation}. \]
Introduce the acoustic (Christoffel) tensor $\Gamma$ of the medium:

$$\Gamma(x, k) U := \varrho(x)^{-1} (C(x) : U \otimes k) k, \quad k \in \mathbb{R}^d, \ U \in \mathbb{R}^n.$$ 

The local balance of momentum for the displacement field $u_\varepsilon$ in $\mathcal{O} \times \mathbb{R}_t$ is the elastic wave equation:

$$\varrho(x) \left( \Gamma(x, i\varepsilon \nabla_x) - (i\varepsilon \partial_t)^2 \right) u_\varepsilon = O(\varepsilon),$$

supplemented with (e.g. Neumann or Dirichlet) boundary conditions on $\partial\mathcal{O} \times \mathbb{R}_t$. 
Some properties of $\Gamma$

- $\Gamma$ is symmetric, real, positive definite (in $\mathcal{O} \times \mathbb{R}^d \setminus \{k = 0\}$): it has $n$ (possibly $r_\alpha$-multiple) positive eigenvalues $\lambda^2_\alpha$ such that

$$\lambda_\alpha(x, k) = c_\alpha(x, \hat{k})|k|, \quad 1 \leq \alpha \leq n,$$

where $\hat{k} = \frac{k}{|k|} \in \mathbb{S}^{d-1}$, and the real eigenvectors $b_\alpha(x, k) = b_\alpha(x, \hat{k})$ can be orthonormalized,

$$\Gamma = \sum_{\alpha=1}^{n} \lambda^2_\alpha b_\alpha \otimes b_\alpha, \quad I_n = \sum_{\alpha=1}^{n} b_\alpha \otimes b_\alpha.$$

- **Example**: three-dimensional isotropic medium ($n = d$),

$$\Gamma(x, \hat{k}) = c_P^2(x)\hat{k} \otimes \hat{k} + c_S^2(x)(I_d - \hat{k} \otimes \hat{k}).$$
The energy density (real, positive) and the power flow density (vector) are:

\[ E_\varepsilon (x, t) = \frac{1}{2} (\rho |\partial_t u_\varepsilon|^2 + \sigma_\varepsilon : \varepsilon_\varepsilon) = T_\varepsilon + U_\varepsilon , \]
\[ \pi_\varepsilon (x, t) = -\sigma_\varepsilon \partial_t u_\varepsilon . \]

They satisfy for all fixed \( \varepsilon \in ]0, \varepsilon_0] \) a conservation equation:

\[ \partial_t E_\varepsilon + \text{div} \pi_\varepsilon (\pi_{\text{dis}}) = 0 (\pi_{\text{inj}}) . \]

What happens when \( \varepsilon \to 0 ? \)
Existing approaches for structural-acoustics

**Statistical Energy Analysis (SEA)**

Westphal 1957, Lyon-Maidanik 1962, Smith 1962, Maidanik 1976 ...

**Vibrational conductivity analogy (VCA)**

Existing approaches for structural-acoustics

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Ray methods 1/3

Principles

• $u_\varepsilon$ is sought for as a WKBJ ansatz:

$$u_\varepsilon(x, t) \simeq e^{i S(x, t)} \sum_{k=0}^{\infty} \varepsilon^k U_k(x, t).$$

• **Eulerian** point of view (b): the phase $S$ satisfies an eikonal equation and the densities $|U_k|^2$ satisfy transport equations.

• **Lagrangian** point of view (c): $(x, \nabla_x S)$ is given as the solution of the associated Hamiltonian system (ray tracing).
Plugging the WKBJ ansatz into the elastic wave equation:

\[ H(s, \nabla_s S)U_0 = 0, \quad \text{[eikonal]} \]
\[ \nabla_s \cdot \left( U_0^T \nabla_\xi H(s, \nabla_s S)U_0 \right) = 0, \quad \text{[0^{th}-order transport]} \]

where:

\[ H(s, \xi) = \varrho(x) \left( \Gamma(x, k) - \omega^2 I_n \right) \]
\[ = \varrho(x) \sum_{\alpha=1}^n \left( \lambda_\alpha^2(x, k) - \omega^2 \right) b_\alpha \otimes b_\alpha \]

is the dispersion matrix of the medium, \( s = (x, t) \in \mathcal{O} \times \mathbb{R}, \xi = (k, \omega) \in \mathbb{R}^d \times \mathbb{R}. \)

Thus \( \mathcal{H} = \det H = 0 = \prod_{\alpha=1}^n \mathcal{H}_\alpha \) where \( \mathcal{H}_\alpha = \varrho \left( \lambda_\alpha^2 - \omega^2 \right), \) and therefore:

\[ \mathcal{H}_\alpha(s, \nabla_s S) = 0. \]
Solving the eikonal equation for $S \in \mathbb{R}$ by the method of characteristics, $(s, \nabla_s S)$ is given as a solution of the system:

$$
\begin{align*}
\frac{ds}{d\tau} &= \nabla_\xi \mathcal{H}_\alpha(s(\tau), \xi(\tau)), & s(0) = s_0, \\
\frac{d\xi}{d\tau} &= -\nabla_s \mathcal{H}_\alpha(s(\tau), \xi(\tau)), & \xi(0) = \xi_0 \neq 0,
\end{align*}
$$

in $T^*(\mathcal{O} \times \mathbb{R}) \setminus \{(s, \xi); \xi = 0\}$, with $\xi_0 = \nabla_s S(s_0)$.

It follows immediately that $\mathcal{H}_\alpha(s(\tau), \xi(\tau)) = \mathcal{H}_\alpha(s_0, \xi_0) = \text{C}^\text{st}$ along the bicharacteristic curves $\tau \mapsto (s(\tau), \xi(\tau))$ solving the above system.

The rays are defined as the projections on $\mathcal{O} \times \mathbb{R}_t$ of these curves, that is $\tau \mapsto s(\tau)$. 

Research program

- **SEA, VCA**: approximations (provably wrong);
- **Ray methods**: superposition principle, regularity... and they give only one particular construction of oscillating solutions.

**Kinetic models**: all oscillating solutions

The **key idea**: phase space description

(which also accounts for the directions in which waves propagate)

- Mechanical modeling for aerospace structures: material heterogeneity and damping, boundary conditions, higher-order kinematics;
- Numerical simulations;
- (Further modeling: diffusion limits).
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Consider the bicharacteristic strip \( \tau \mapsto (s(\tau), \xi(\tau)) \) as the paths in phase space of some energy “particles” of which density is denoted by \( W(s(\tau), \xi(\tau)) \), the latter shall satisfy:

\[
\frac{dW}{d\tau} = \{W, H\} = 0 ,
\]

[TE]

where \( \{f, g\} = \nabla_\xi f \cdot \nabla_s g - \nabla_s f \cdot \nabla_\xi g \) (Poisson bracket).

But \( \frac{d\omega}{d\tau} = 0 \) and \( H = \sum H_\alpha b_\alpha \otimes b_\alpha \) with \( \mathbb{I}_n = \sum b_\alpha \otimes b_\alpha \), thus:

\[
W = \sum_{\alpha=1}^{n} w_\alpha \delta(H_\alpha) .
\]

The link between \( W \) and the sequence \( (u_\varepsilon) \) is established by the Wigner measure of the latter.
Kinetic modeling
Irreversible system

- GENERIC framework:
  \[
  \frac{dW}{d\tau} = \{W, \mathcal{H}\} + [W, S] = 0 ,
  \]

  where \( S \) is the entropy and \([S, W]\) is the dissipative bracket, satisfying the degeneracy conditions:
  \[
  \{S, W\} = [\mathcal{H}, W] = 0 .
  \]

- The link between \( W \) and the sequence \((u_\varepsilon)\) is established by the Wigner measure of the latter considering the wave equation with a time-dependent acoustic tensor.

Grmela-Öttinger PRE 56(6) 6620, 1997
Öttinger-Grmela PRE 56(6) 6633, 1997
Why quadratic observables?

- Let:

\[ u_\varepsilon(x) = m(x) + \sigma(x) \sin \frac{x}{\varepsilon}, \]

then \((u_\varepsilon) \rightharpoonup m\) weakly in \(L^2(\mathbb{R})\) as \(\varepsilon \to 0\), but \((u_\varepsilon)\) has no strong limit in any \(L^p\).
Why quadratic observables?

Now for any observable $\varphi \in C_0(\mathbb{R})$:

$$\lim_{\varepsilon \to 0} (\varphi(x)u_\varepsilon, u_\varepsilon)_{L^2} = \int_{\mathbb{R}} \varphi(x) \left( (m(x))^2 + \frac{1}{2} (\sigma(x))^2 \right) \, dx.$$
Why quadratic observables?

- Take an observable of the form:

\[
\varphi(x, \partial_x) u_\varepsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik \cdot x} \varphi(x, ik) \hat{u}_\varepsilon(k) \, dk.
\]
Why quadratic observables?

- Take an observable of the form:

\[
\varphi(x, \varepsilon \partial_x) u_\varepsilon(u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik \cdot x} \varphi(x, i\varepsilon k) \hat{u}_\varepsilon(k) dk.
\]
Why quadratic observables?

Then:

$$\lim_{\varepsilon \to 0} (\varphi(x, \varepsilon \partial_x) u_\varepsilon, u_\varepsilon)_{L^2} = \iint_{\mathbb{R}^2} \varphi(x, ik) W[u_\varepsilon](dx, dk),$$

where $W[u_\varepsilon]$ is the (positive) Wigner measure of $(u_\varepsilon)$. 
Example: the overall kinetic and strain energy densities are in the high-frequency limit $\varepsilon \to 0$

$$
\lim_{\varepsilon \to 0} T_\varepsilon(x, t) = \frac{1}{2} \varrho(x) \int_{\mathbb{R}^d} \text{Tr} \ W[\varepsilon \partial_t u_\varepsilon(\cdot, t)](x, dk),
$$

$$
\lim_{\varepsilon \to 0} U_\varepsilon(x, t) = \frac{1}{2} \varrho(x) \int_{\mathbb{R}^d} \Gamma(x, k) : W[u_\varepsilon(\cdot, t)](x, dk),
$$

(eventually one can prove that they are equal in this very limit), or

$$
\lim_{\varepsilon \to 0} E_\varepsilon(x, t) = \sum_{\alpha=1}^{n} \int_{\mathbb{R}^d} w_\alpha(x, t; dk).
$$
Boundary conditions

Rankine-Hugoniot condition

- Let a **discontinuity front** $\Sigma_D$ be defined by the hypersurface in phase space:

$$\Sigma_D = \{(s, \xi) \in T^* (\mathcal{O} \times \mathbb{R}); \Sigma(s, \xi) = 0\}.$$  

- The **Rankine-Hugoniot condition** associated to piecewise continuous (weak) solutions of the Liouville equation [TE] on $\Sigma_D \cap \{\mathcal{H}(s, \xi) = 0\}$:

$$[[\{\mathcal{H}, \Sigma\} W]] = 0.$$

- **Consequence**: continuity of the overall normal power flow across any fixed interface $\{\Sigma(x, k) = 0\}$,

$$\sum_{\alpha=1}^{n} [[\{\lambda_\alpha, \Sigma\} w_\alpha]] = 0.$$
Boundary conditions
Snell-Descartes law

- The Hamiltonian $\mathcal{H} = 0$ across $\Sigma_D$ at $t_0$ such that $\Sigma(s(t_0), \xi(t_0)) = 0$:
  \[
  \lambda_\alpha(x(t_0^-), k(t_0^-)) = \lambda_\beta(x(t_0^+), k(t_0^+)), \quad 1 \leq \alpha, \beta \leq n.
  \]

- **Consequence:** Snell-Descartes law of diffraction,
  \[
  P_x k = \text{constant},
  \]
  where $P_x = I_d - \hat{n}(x) \otimes \hat{n}(x)$ is the orthogonal projection on the tangent plane to a fixed interface at $x$.  

Example: acoustic waves $\lambda_1(x, k) = \pm c(x)|k|$ ($n = 1$).

Convex acoustic interface with $c_+ > c_-$

Concave acoustic interface with $c_+ > c_-$

Jin-Yin JCP 227(12) 6106, 2008
Boundary conditions
Reflection/transmission coefficients

- \(\{\mathcal{H}, \Sigma\} > 0\): transverse reflections (the hyperbolic set);
- \(\{\mathcal{H}, \Sigma\} < 0\): total reflections (the elliptic set);
- \(\{\mathcal{H}, \Sigma\} = 0\): critical reflections (the glancing set):
  - \(\{\mathcal{H}, \{\mathcal{H}, \Sigma\}\} > 0\): diffractive rays,
  - \(\{\mathcal{H}, \{\mathcal{H}, \Sigma\}\} < 0\): gliding rays,
  - \(\{\mathcal{H}, \{\mathcal{H}, \Sigma\}\} = 0\): higher-order gliding rays (w. inflection point).

For **elastic waves** the situation is a mix of all cases!

Boundary conditions raise profound mathematical issues, mostly unsolved to date. Unfortunately they have also great engineering relevance!
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Discontinuous finite elements for the TE 1/2

- TE is a **linear conservation equation** in phase space $\Omega = \mathcal{O} \times \mathbb{R}^d \ni z = (x, k)$:

$$\mathcal{L} w_\alpha \equiv \partial_t w_\alpha + \nabla_z \cdot \mathcal{F}(w_\alpha) = 0, \quad \mathcal{F}(w_\alpha) = \begin{pmatrix} \nabla_k \lambda_\alpha \\ -\nabla_x \lambda_\alpha \end{pmatrix} w_\alpha;$$

- Initial conditions: $w_\alpha(dz; t = 0) = w^0_\alpha(dz)$;
- The modes $\alpha$ get coupled by the reflection/transmission conditions.
Consider the partition $\mathcal{T}_h = \bigcup_{r=1}^{N} \Omega_r$ of $\Omega$ into $N$ non-overlapping subdomains, or finite elements, and the trial space $\mathcal{W}_h$ of piecewise continuous functions on $\mathcal{T}_h$.

Variational formulation using the test space $\mathcal{V}_h$:

"Weak" form: Find $w_{\alpha} \in \mathcal{W}_h$ s.t.

$$\int_{\Omega_r} \left( v \partial_t w_{\alpha} - \nabla z v \cdot F(w_{\alpha}) \right) \, d\Omega + \int_{\partial\Omega_r} v F^*_r(w_{\alpha}^-, w_{\beta}^+) \cdot \mathbf{n}_r \, d\gamma(z) = 0, \quad \forall v \in \mathcal{V}_h,$$

where $F(w_{\alpha}) \cdot \mathbf{n}|_{\partial\Omega_r}$ is replaced by $F^*_r(w_{\alpha}^-, w_{\beta}^+) \cdot \mathbf{n}_r$ because $w_{\alpha}$ may have two different values on both sides of the edge $\partial\Omega_r$. 
Consider the partition $\mathcal{T}_h = \bigcup_{r=1}^{N} \Omega_r$ of $\Omega$ into $N$ non-overlapping subdomains, or finite elements, and the trial space $\mathcal{V}_h$ of piecewise continuous functions on $\mathcal{T}_h$.

Variational formulation using the test space $\mathcal{V}_h$:

"Strong" form (Lesaint & Raviart 1974): Find $w_\alpha \in \mathcal{V}_h$ s.t.

$$\int_{\Omega_r} v \mathcal{L} w_\alpha \, d\Omega = \int_{\partial\Omega_r} v \left( \mathcal{F}(w^-_\alpha) - \mathcal{F}^*(w^-_\alpha, w^+_\beta) \right) \cdot \hat{n}_r \, d\gamma(z), \quad \forall v \in \mathcal{V}_h,$$

e.g. penalization methods.
Discontinuous finite elements for the TE 2/2

- Consider the partition $\mathcal{T}_h = \bigcup_{r=1}^{N} \Omega_r$ of $\Omega$ into $N$ non-overlapping subdomains, or finite elements, and the trial space $\mathcal{W}_h$ of piecewise continuous functions on $\mathcal{T}_h$.
- Variational formulation using the test space $\mathcal{V}_h$:

  "Ultra-weak variational formulation":

$$\partial_t \left( \int_{\Omega_r} v w_\alpha \, d\Omega \right) + \int_{\partial \Omega_r} \mathcal{F}_r^* (w^-_\alpha, w^+_\beta) \cdot \hat{n}_r \, d\gamma(z) = 0, \quad \forall v \in \mathcal{V}_h,$$

choosing $\mathcal{V}_h = \{ v; \mathcal{L}^* v = 0 \text{ in } \Omega_r \}$ (plane waves for example are $v(z,t) = V \phi(k \cdot x - \omega t)$ such that $\Gamma(z) V = \omega V$, with the acoustic tensor $\Gamma(z) := \text{diag}\{\lambda_\alpha(z)\}_{1 \leq \alpha \leq n}$).

Després CRAS I 318 939, 1994


Example #1: beam junction

Energy transport in a beam junction with \( \phi = 60^\circ \), \( \frac{E_2}{E_1} = 2 \), \( \nu_1 = \nu_2 = 0.3 \), and \( T = \frac{L}{c_{T_2}} \).
**Example #1: beam junction**

Energy transport in a beam junction with $\phi = 60^\circ$, $\frac{E_2}{E_1} = 2$, $\nu_1 = \nu_2 = 0.3$, and $T = \frac{L}{c_{T2}}$. 

Runge-Kutta discontinuous FEM w. Gauss-Lobatto nodal expansion
$N^c = 40$, $N = 10$, RK-SSP(8, 8)

— unfiltered
— 0th-order exponential filter
Example #1: beam junction

"Hat" source: Legendre modal expansion
\( N = 40, N = 5, \text{RK-SSP}(5, 4) \)

"Square" source: nodal expansion
\( N = 40, N = 5, \text{RK-SSP}(5, 4) \)

Energy ratios \( t \mapsto \frac{\mathcal{E}_P(t)}{\mathcal{E}_T(t)} \) within each sub-structure (beam #1, beam #2); \( \phi = 60^\circ \),
\[
\frac{E_2}{E_1} = 2, \; \nu_1 = 0.4, \; \nu_2 = 0.1, \; T = \frac{L}{c_{T2}}.
\]

Savin PEM 28 194, 2012
Example #2: beam truss

Runge-Kutta discontinuous FEM
w. Legendre modal expansion
\[ \mathcal{N} = 1212, \ N = 8, \ \text{RK-SSP}(8,8) \ (\text{CFL} = 10^{-2}) \]

Energy transport in a beam truss with \( T = \frac{L}{c_T} \).

\[ \frac{\mathcal{E}_T^r(t)}{\mathcal{E}_P^r(t)} \quad \xrightarrow{t \to +\infty} \quad \frac{2c_P}{c_T} \quad (1 \leq r \leq 4) \]
Example #3: thick plate

Runge-Kutta discontinuous FEM w. mixed nodal/modal expansion
$\mathcal{N} = 21 \times 21$, $N = 5$ (LGL nodes), $P = 2$ (Fourier), SSP(3, 3)

Energy transport in a thick plate with $\nu = 0.3$ and $T = \frac{L}{c_S}$.

Issues with Fourier expansion: dispersion, positivity.
Example #3: thick plate

Energy transport in a thick plate with $\nu = 0.3$ and $T = \frac{L}{c_s}$.

Issue with Lagrange expansion: ray effect.
Example #4: transport in a shell junction

Direct Monte-Carlo method
10^6 sample paths

Energy transport in a shell junction impacted by a shear load with $\phi = 15^\circ$, $\frac{E_2}{E_1} = 2$, $\nu_1 = 0.3$, $\nu_2 = 0.2$, $T = \frac{L}{c_{T_2}}$.
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Wave propagation in heterogeneous media

- **Scales:** $\lambda$ – wavelength, $\ell$ – correlation length, $L$ – propagation/observation distance...

... and the **mesoscopic scale** $\ell_{sc}$ – scattering mean free path.

Ryzhik HFWP’05, 2005
Radiative transfer in anisotropic media

Example: triclinic random medium

Velocity field in anisotropic heterogeneous half-space with mean homogeneous isotropy, monopole on the surface; \( c_P = 2000 \text{ m/s}, \ c_S = 1000 \text{ m/s}, \ \rho = 2000 \text{ kg/m}^3 \).
Radiative transfer in anisotropic media

Example: anisotropic diffusion

Privileged directions for diffusion corresponding to the principal axes of diffusion

$\rightarrow$ localisation when $\lambda \sim \ell_{\text{sc}}$?

Lobkis-Weaver JASA 124(6) 3528, 2008
Perspectives

Modeling

- Diffraction and glancing;
- Diffusive regimes for slender structures $\rightarrow$ SEA;
- "Multi-physics" coupling with quadratic quantities.

Numerical simulations

- Regularized interface conditions by diffusion (Dafermos 1977, Fornet 2007);
- Discontinuous finite elements: phase-space discretization, positivity, filtering, goal-oriented adaptive strategies...
- "Multi-scale" coupling of transport & diffusion.
Hot topics

- RTN HYKE 2002–2005, GdR CHANT, GdR Mésolm®ge, GdR Ondes, GdR Ultrasons, ERC StG NuSiKiMo, ANR MNEC...
- CIRM’05, HFWP’05, Waves’07, Waves’09, HYKE’10, Waves’11, HYP’12, Euromech 540...
- AA, CiCP, CiMP, CiMS, JASA, JCP, JHDE, JMP, JSC, KRM, WM, WRCM...