THE UNIVERSE ACCORDING TO LEIBNIZ,

THE COUNTABLE RANDOM GRAPH

AND

ULTRAHOMOGENEOUS COMBINATORIAL STRUCTURES

LEIBNIZ 1697

"De rerum originatione radicale"
(On the radical originatcon of things)

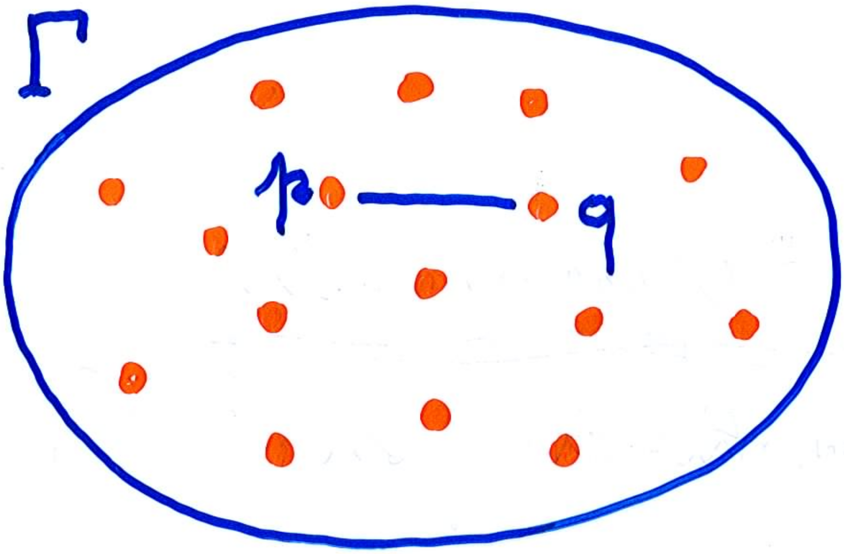
metaphysical statements:

- The real world is the best of all possible worlds

"best" ≡ maximizes
- symmetry (beauty)
- variety of substructures

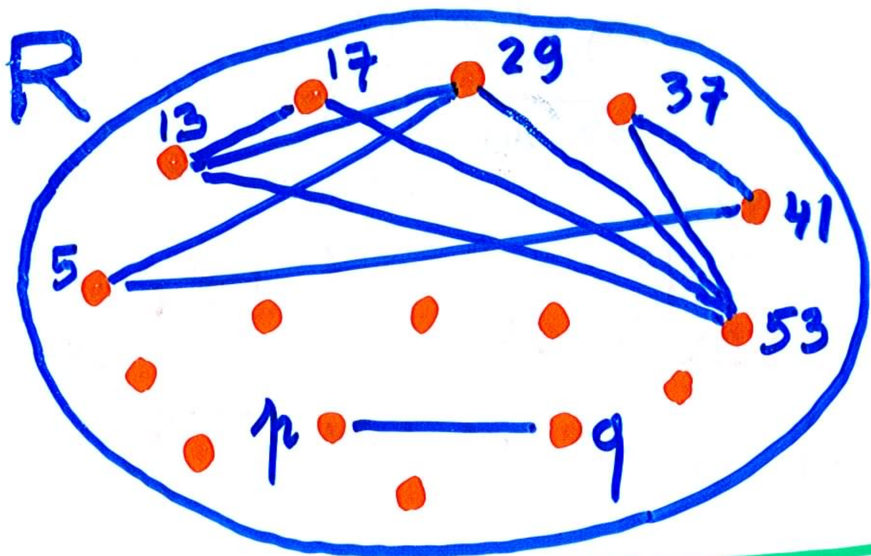
- The best of all possible worlds is also the most probable random structure

THE COUNTABLE RANDOM GRAPH



$\aleph_0$ vertices

For each pair $\{p, q\}$ of vertices of Γ, toss a coin to decide whether or not p and q are joined by an edge



vertices: prime numbers $\equiv 1 \pmod 4$

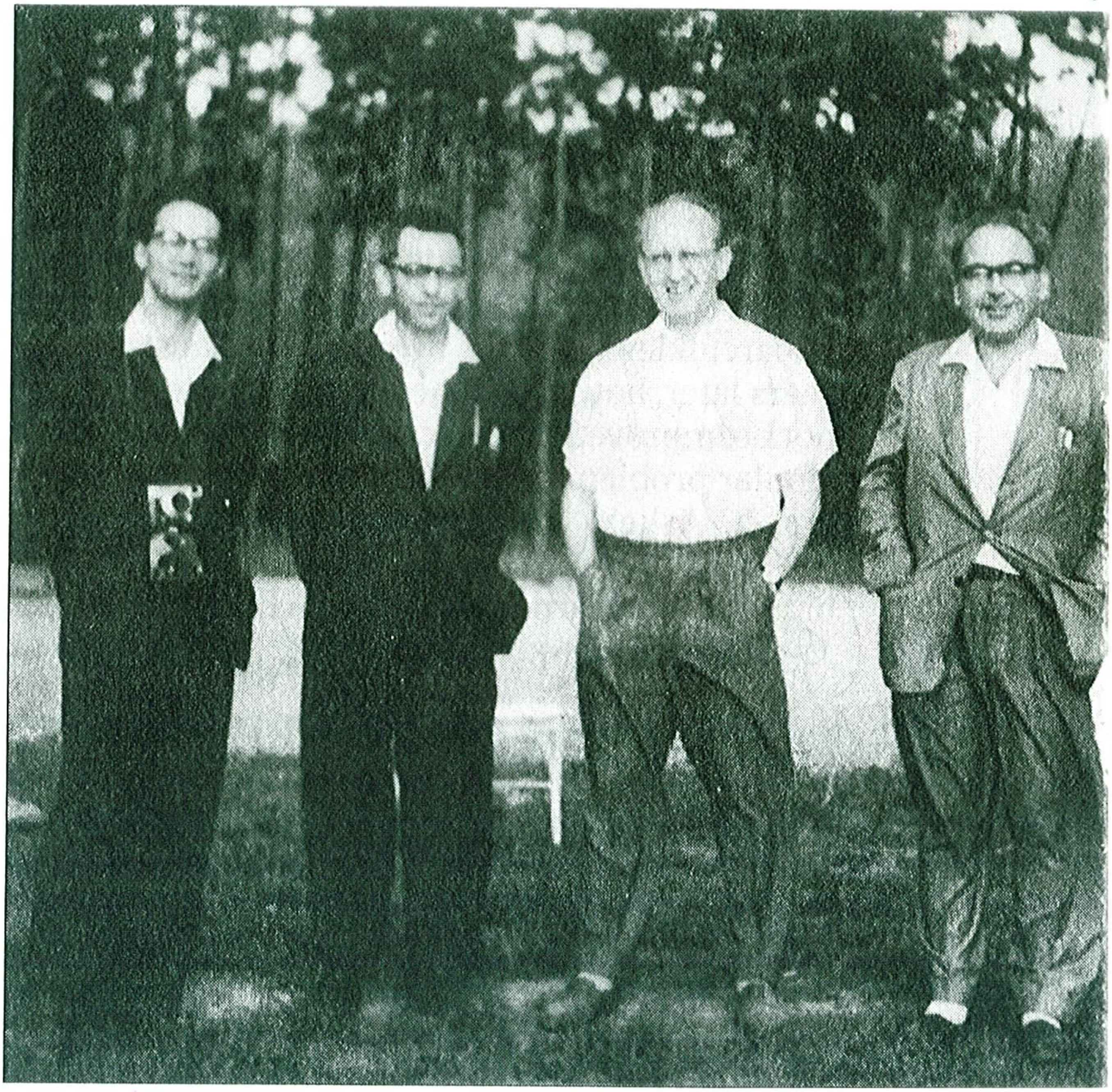
$$\boxed{p \sim q \iff \left(\frac{p}{q}\right) = 1}$$

GAUSS: $p \equiv q \equiv 1 \pmod 4 \Rightarrow \left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$

$p \sim q \Rightarrow q \sim p$ (undirected graph)

Paul
ERDÖS

Alfred
RÉNYI



THEOREM (ERDÖS and RÉNYI 1963)

$$\mathbb{P}(\Gamma \cong R) = 1 \quad !!!$$

R = the countable random graph

R has many extraordinary and unbelievable properties:

- R is universal: every finite or countable graph appears in R as an induced subgraph!
- R is indestructible: if the vertex set of R is partitioned into finitely many parts, the induced subgraph on one of these parts is $\cong R$!
- R has many symmetries:

$$|\mathrm{Aut}\, R| = 2^{\aleph_0} \; !$$

Aut R is a simple group

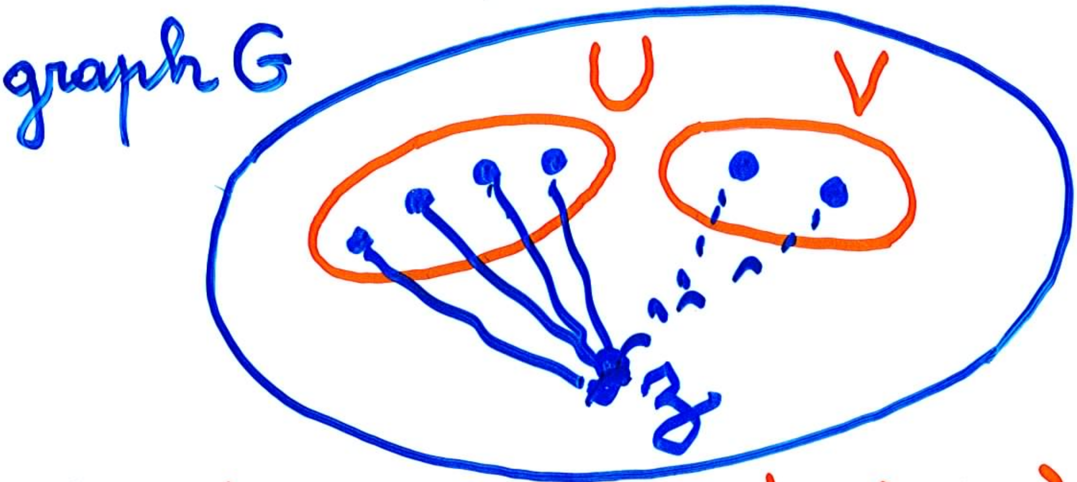
R is ultrahomogeneous!

- R is isomorphic to its complement

etc, etc .....

Proof of the Theorem

PROPERTY (*):



U, V = any two disjoint finite sets of vertices of G

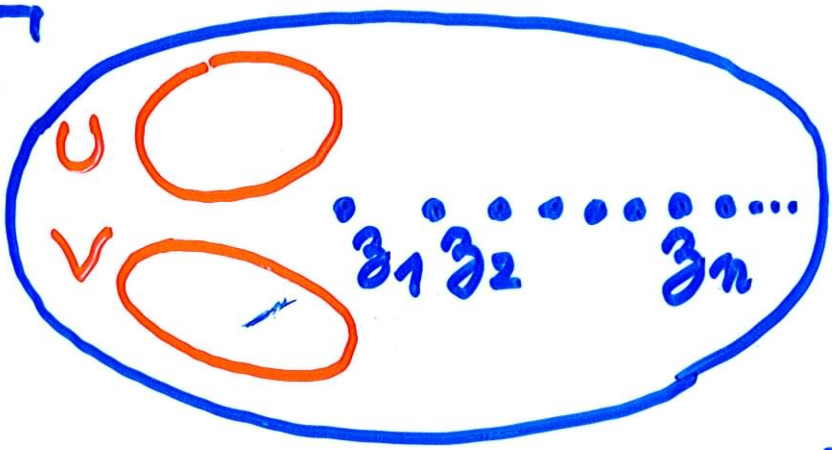
$$\Rightarrow \exists \text{ vertex } z \begin{cases} \sim \text{ every } u \in U \\ \not\sim \text{ every } v \in V \end{cases}$$

① $\mathbb{P}(\Gamma \text{ has property } (*)) = 1$

② R has property (*)

③ Two countable graphs having property (*) are necessarily isomorphic

1 Γ



$|U \cup V| = k$

$$\mathbb{P}(z_i \text{ good}) = \left(\tfrac{1}{2}\right)^k$$

$$\Rightarrow \mathbb{P}(z_i \text{ bad}) = 1 - \left(\tfrac{1}{2}\right)^k$$

$$\Rightarrow \mathbb{P}(z_1, \dots, z_n \text{ all bad}) = \left(1 - \left(\tfrac{1}{2}\right)^k\right)^n \xrightarrow{n \to \infty} 0$$

$$\Rightarrow \mathbb{P}(\text{all } z_i\text{'s bad}) = 0$$

② $U = \{p_1, \dots, p_m\}$

$V = \{q_1, \dots, q_n\}$

Choose a_i such that $\left(\frac{a_i}{p_i}\right) = 1$

b_j such that $\left(\frac{b_j}{q_j}\right) = -1$

$$\begin{cases} x \equiv 1 \pmod{4} \\ x \equiv a_1 \pmod{p_1} \\ \vdots \\ x \equiv a_m \pmod{p_m} \\ x \equiv b_1 \pmod{q_1} \\ \vdots \\ x \equiv b_n \pmod{q_n} \end{cases}$$

has a unique solution

$x \equiv x_0 \pmod{4 p_1 \cdots p_m q_1 \cdots q_n}$

But $\gcd(x_0, 4 p_1 \cdots p_m q_1 \cdots q_n) = 1$

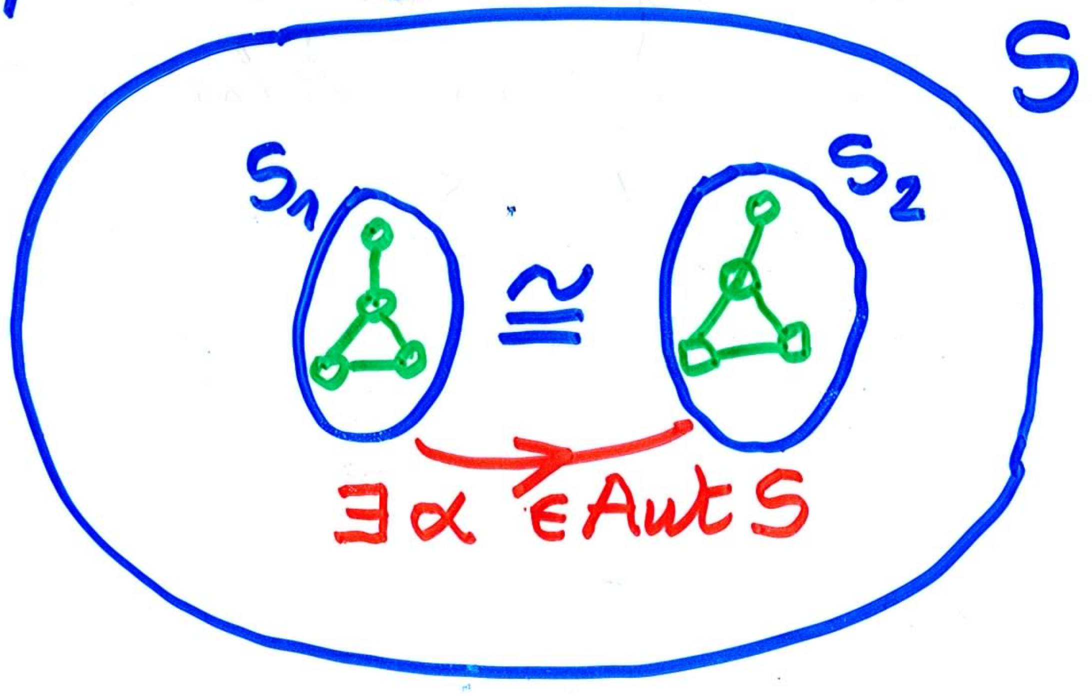
So, by DIRICHLET's theorem:

$\exists$ prime p solution of this congruence

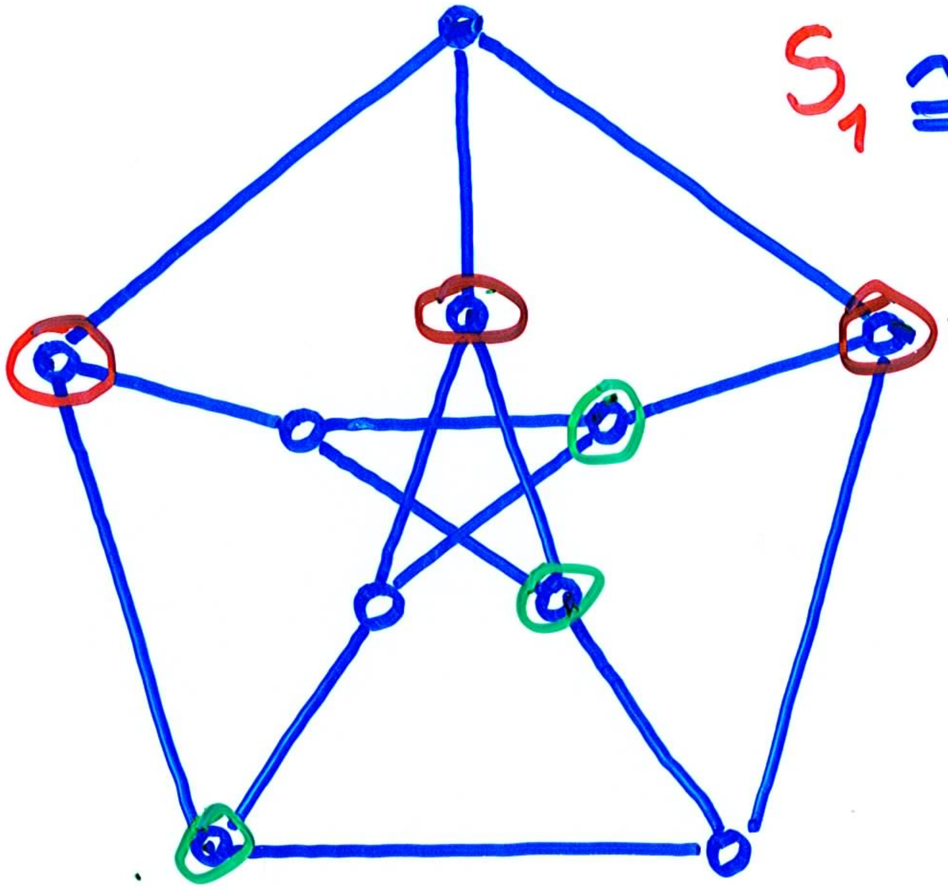
- $p \equiv 1 \pmod{4} \Longrightarrow p \in R$
- $p \equiv a_i \pmod{p_i} \Longrightarrow p \sim p_i \in U$
- $p \equiv b_j \pmod{q_j} \Longrightarrow p \nsim q_j \in V$

S = set carrying some structure (graph, metric space, poset, ...)

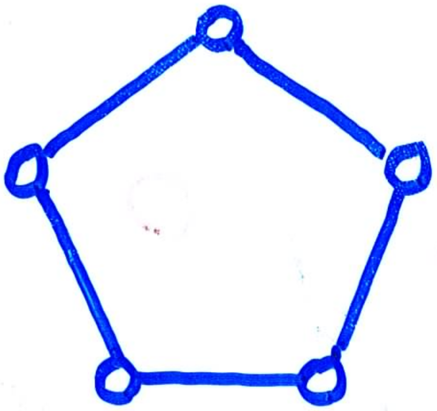
- S is **homogeneous** if, whenever the structures induced on two **FINITE** subsets S_1 and S_2 are isomorphic, there is **some** automorphism of S mapping S_1 onto S_2

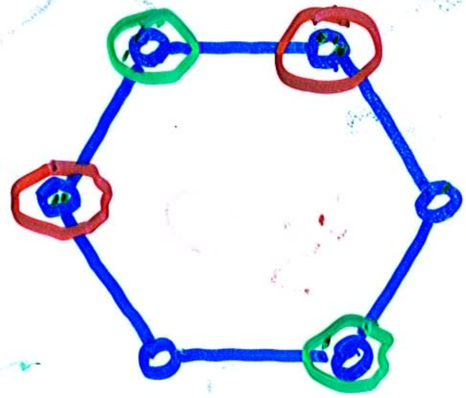


- S is **ultrahomogeneous** if **every** isomorphism from S_1 to S_2 extends to an automorphism of S



$S_1 \cong S_2$

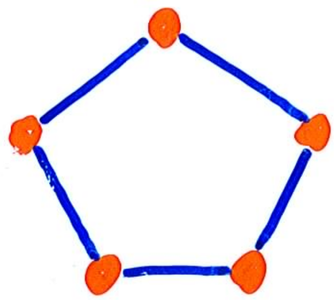




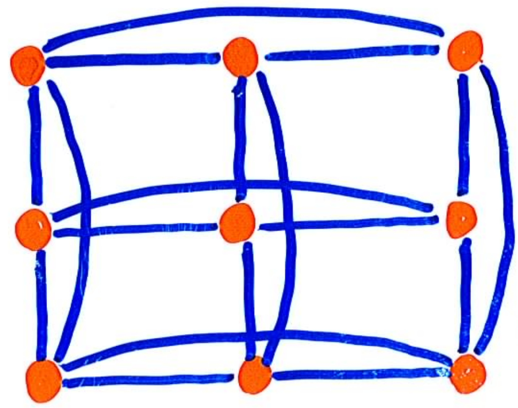
THEOREM (SHEEHAN 1974, GARDINER 1976, GOLFAND and KLIN 1978)

Every finite ultrahom graph is one of the following:

1. a disjoint union of isomorphic complete graphs
2. the complement of a graph in 1
3. the pentagon



4. the 3x3 grid



THEOREM (RONSE 1978)

The list of finite homogeneous graphs is exactly the same

COUNTABLE ULTRAHOMOGENEOUS GRAPHS were classified in 1980 by LACHLAN and WOODROW:

- 4 countably infinite families (2 trivial, 2 non-trivial)
- the random graph R of Erdös and Rényi

ULTRAHOMOGENEOUS DIRECTED GRAPHS ?

- Finitely many vertices: classified by LACHLAN 1982
- Countably many vertices: classified by CHERLIN 1998 (Memoirs AMS)

$\exists\ 2^{\aleph_0}$ such graphs

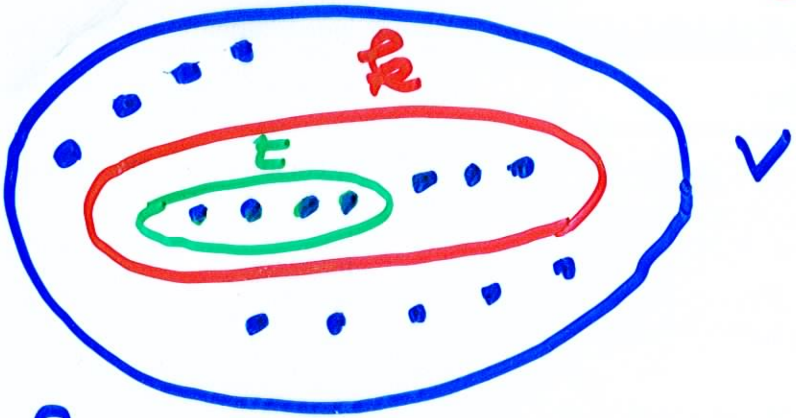
JACOB STEINER

1796 - 1863

STEINER SYSTEMS

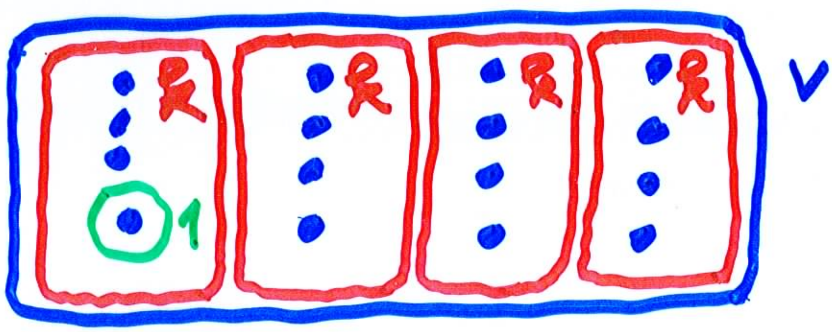
$0 < t < k < v$ integers

Steiner system $S(t, k, v)$
= set of v elements called points
+ collection of k-subsets called blocks such that any set of t points is contained in $\underline{1}$ block.



Examples

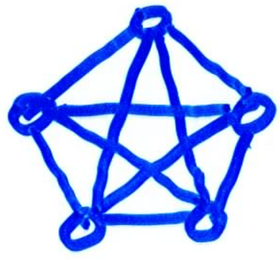
- $S(1, k, v)$



= partition of a v-set into k-subsets

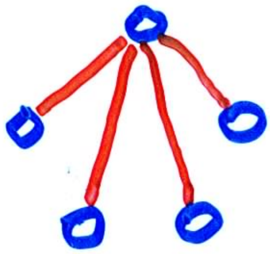
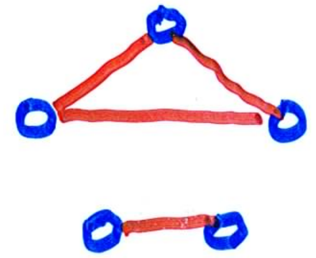
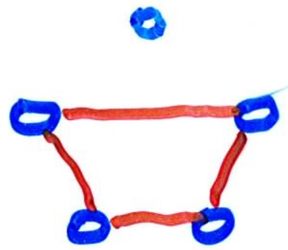
TRIVIAL!

- $S(3,4,10)$



points = edges of K_5

blocks = subgraphs of K_5 of the form

 or  or 

$5 + 10 + 15 = 30$

- $S(5,6,12)$ (Ernst WITT, 1938)

points = elements of $\mathbb{Z}_{11} \cup \{\infty\}$

blocks = all images of $\{1,3,4,5,9,\infty\}$ under the group $PSL(2,11)$

$$PSL(2,11) = \left\{ x \longrightarrow \frac{ax+b}{cx+d} \;\middle|\; a,b,c,d \in \mathbb{Z}_{11} \text{ and } ad-bc = \square \neq 0 \right\}$$

- Lots of examples known for $t \leq 3$
- Only finitely many examples known for $t = 4$ or 5.
- <u>NO</u> example known for $t \geq 6$!

Homogeneous and ultrahomogeneous Steiner systems were classified

- in 1998 for $t = 2$
  (A. DEVILLERS + J. DOYEN in JCT)
- in 2002 for $t \geq 3$
  (A. DEVILLERS in JCD)

<u>THEOREM</u> (DEVILLERS) The only homogeneous $S(t, k, v)$ with $t \geq 4$ are the Witt systems

$S(4,5,11), S(5,6,12), S(4,7,23), S(5,8,24)$

associated with the Mathieu groups

M_{11}, M_{12}, M_{23}, M_{24}

+ with the Golay error-correcting codes
+ with the Leech lattice (densest sphere packing in $\mathbb{R}^{24}$)