Theories of Computation for Continuous Systems. Computing with Analog Models.

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Colloquium Morgenstern
15 March 2012

Menu

Sub-menu

Overall objective

Main objective

Understand computation theories for CONTINUOUS systems.

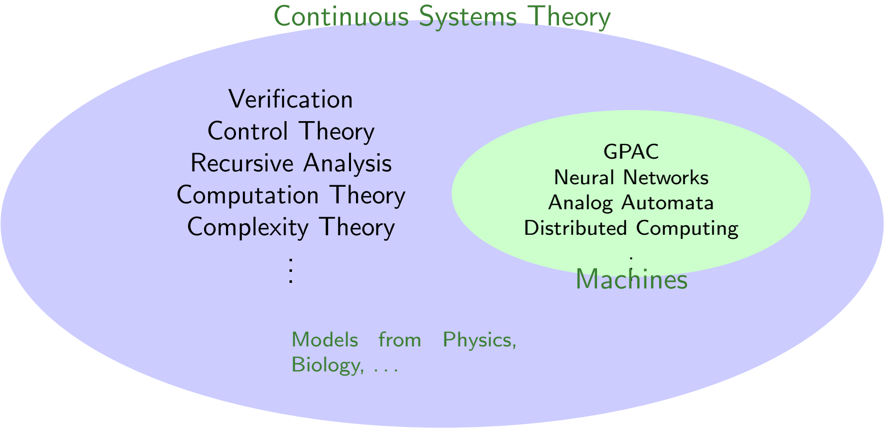
Dynamical Systems
over a continuous space
$\mathcal{H} = (\mathbb{R}^n, f)$

Discrete-Time

$$\left\{ \begin{array}{lcl} y(t+1) & = & f(y(t)) \\ y(0) & = & x \end{array} \right.$$

Continuous-Time

$$\left\{ \begin{array}{lcl} y' & = & f(y(t)) \\ y(0) & = & x \end{array} \right.$$


Continuous Systems Theory
Verification
Control Theory
Recursive Analysis
Computation Theory
Complexity Theory
GPAC
Neural Networks
Analog Automata
Distributed Computing
Machines
Models from Physics, Biology, ...

Continuous Systems Theory

Verification
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Complexity Theory
⋮

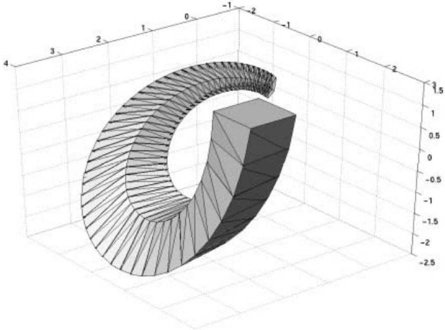
Models from Physics, Biology, ...

Sub-menu

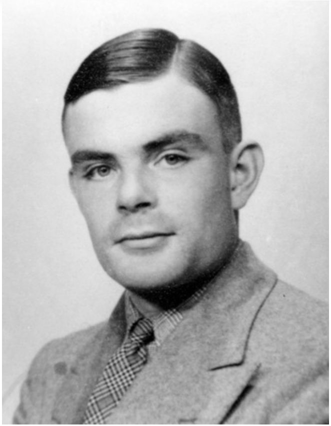
Main Focus

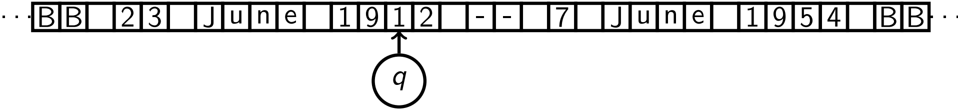
Verification and Control Theory

- **Reachability**. Given $\mathcal{H}$, x_0, $X \subset \mathbb{R}^n$, decide if there is a trajectory going from x_0 to X.
- **Stability**. Given $\mathcal{H}$, decide if all trajectories go to the origin.

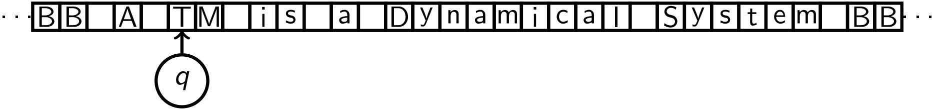


Alan M. Turing





Preliminary: Digital World = Discrete Time and Space



A Turing machine is a particular discrete-time discrete-space dynamical systems.

- A Turing machine over alphabet Σ corresponds to a discrete time dynamical system

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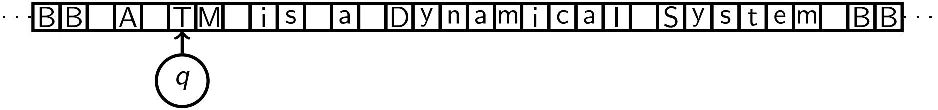

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$$(Q \times \mathbb{N} \times \Sigma^*, \vdash).$$

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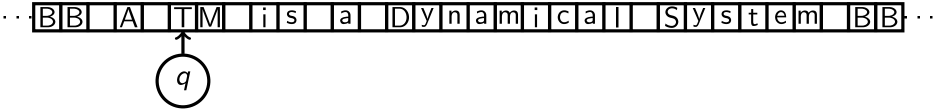

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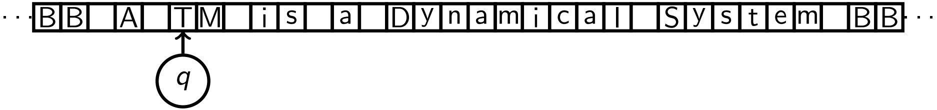

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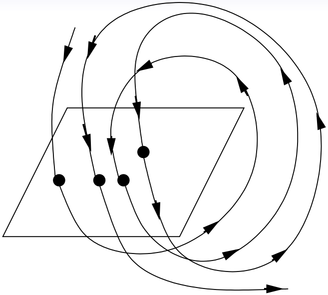
- A Turing machine over alphabet Σ corresponds to a discrete time dynamical system

$$(\mathbb{N}, \vdash).$$

- As $\mathbb{N} \subset \mathbb{R}$, it can be embedded into a continuous space dynamical system

$$(\mathbb{R}^m, f).$$

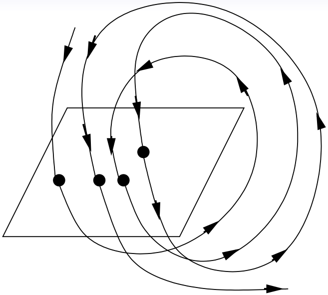
Dynamic Undecidability



Dynamic Undecidability Results:

- [Moore90]
- [Ruohonen93]
- [Siegelmann-Sontag94]
- [Asarin-Maler-Pnueli95]
- [Branicky95]
- [Graça-Campagnolo-Buescu2005]

Dynamic Undecidability



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All non-trivial questions about dynamical systems are hard, from a computability and complexity point of view.

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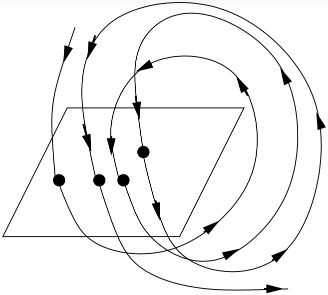

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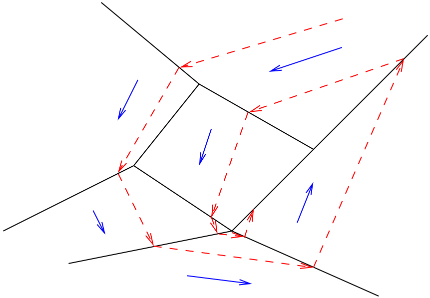
PROBLEMATIC !!

Amazing/EVEN MORE Problematics facts about continuous time systems

- Space contraction.
- Time contraction.
- Zeno's paradox phenomenon.

Piecewise Constant Derivative systems

PCD systems [Asarin-Maler-Pnueli94]:



$$dx/dt = f(x)$$

with $f : \mathbb{R}^d \to \mathbb{Q}^d$ piecewise constant:

1. $Range(f) = C$,

   $\#C < \infty$

2. for all $c \in C$, $f^{-1}(c)$ is a finite union of polyhedral convex subsets of $\mathbb{R}^d$.

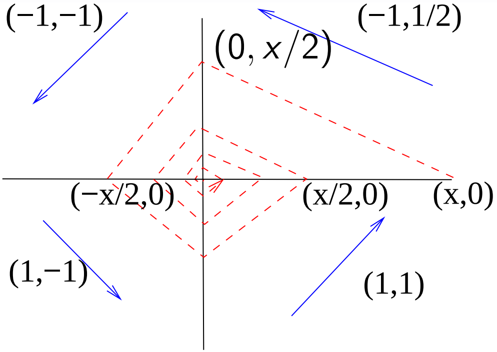
PCD systems in finite discrete time

Discrete time: number of regions crossed.

Theorem:

1. The reachability problem for PCD systems of dimension 2 is decidable [Asarin-Maler-Pnueli95].
2. PCD systems of dimension $d \geq 3$ can simulate Turing machines [Asarin-Maler-Pnueli95].

A trajectory of a PCD system



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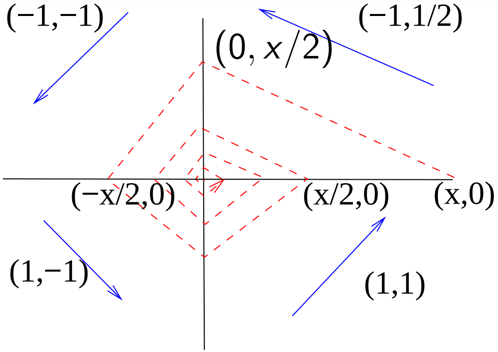

$$5/2(x + x/2 + x/4 + \dots) = 5$$

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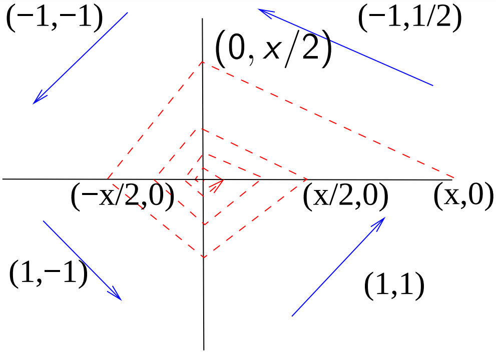
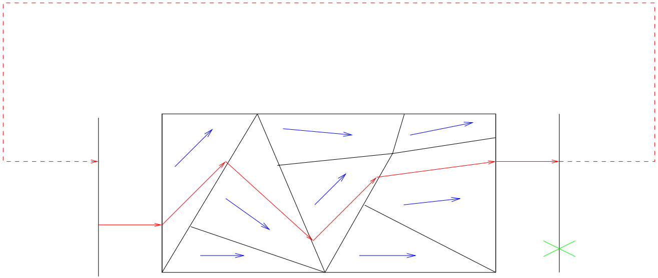


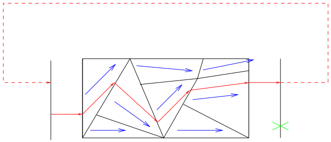
$$5/2(x + x/2 + x/4 + \dots) = 5$$

Observation [Zeno -490/-425]: to a finite continuous time can correspond a transfinite discrete time.

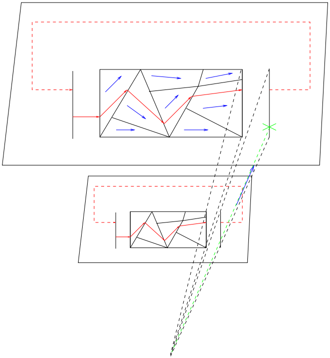
IDEA: Abstract 2-dimensional Representation of a 3-dimensional Turing Machine



IDEA: The Same But With Dimensions Divided by 2



IDEA: Recognizing the Halting Problem of a Turing Machine in dimension 4



In continuous time? [Bournez99]

Continuous time: time taken by the trajectory.

Dimension	Languages semi-recognized
2	$< \Sigma_1$
3	Σ_1
4	Σ_2
5	Σ_ω
6	$\Sigma_{\omega+1}$
7	Σ_{ω^2}
8	Σ_{ω^2+1}
...	...
2p+1	$\Sigma_{\omega^{p-1}}$
2p+2	$\Sigma_{\omega^{p-1}+1}$

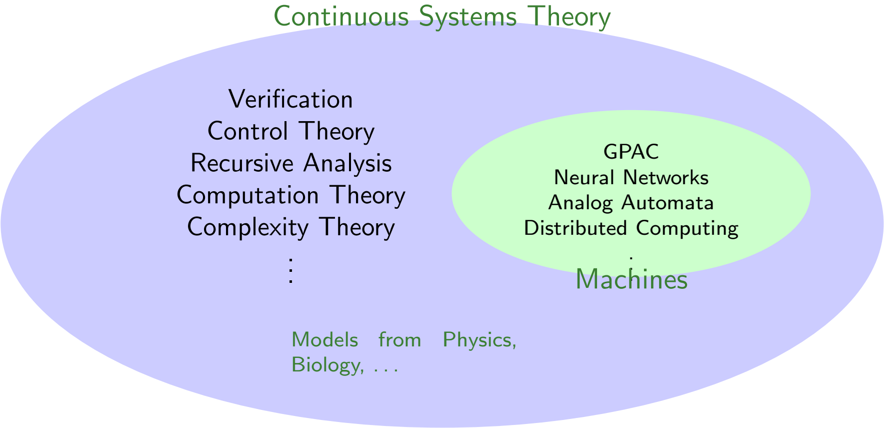
Extending [Asarin-Maler95].

Smooth version

[Ruohonen97]: Space and time contractions can be used to prove that systems $(\mathbb{R}^m, f)$, with f smooth (i.e. $\mathcal{C}^\infty$), on a compact finite-dimensional domain, can simulate arbitrary Turing machines.

[Moore98] conjecture: No analytic function on a compact, finite-dimensional space, can simulate a Turing machine trough a reasonable input and output encoding.

Sub-menu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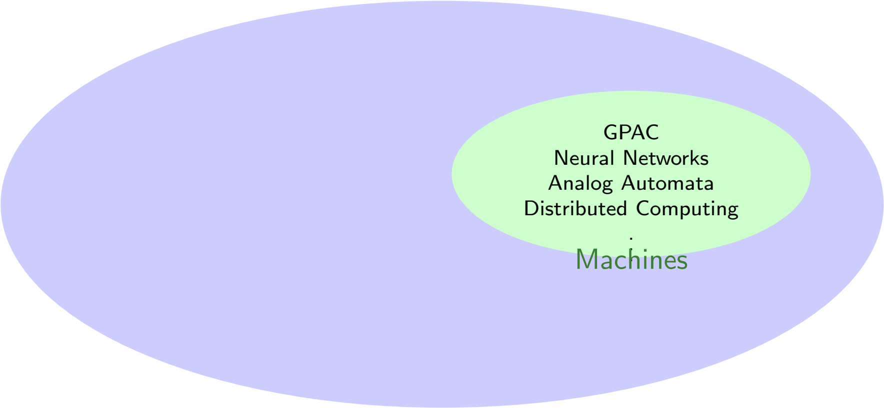
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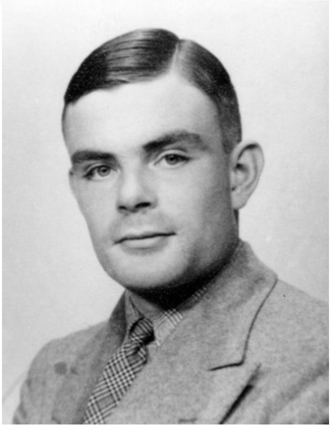
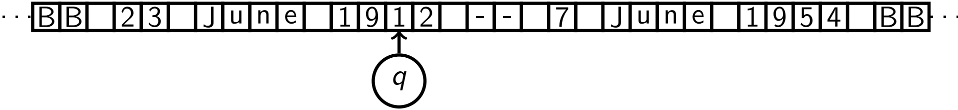
Alan M. Turing





The digital world

- Many models of computations:
  - Recursive functions, Kurt Gödel, 1931-34.
  - Turing machines, Alan Turing, 1936.
  - λ-calculus, Alonzo Church, 1936.
  - Post systems
  - ...
- But, equivalent
  - at the computability level, through Church Turing's thesis
  - and also roughly equivalent at the complexity level: P, NP, ...

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- But, equivalent
  - at the computability level, through Church Turing's thesis
  - and also roughly equivalent at the complexity level: P, NP, ...
- These are digital models: time is discrete, space is discrete.

What about analog models?

The (digital) Picture

Church Thesis	“What is effectively calculable is computable”
Thesis M	“What can be calculated by a machine is computable”
Thesis?	“What can be calculated by a model is computable”

(following [Copeland2002])

Understanding computational power of models helps to understand

- limits of mechanical reasoning.
- limits of machines.
- limits of models.

Menu

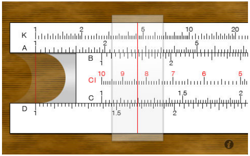
Sub-menu

Some Analog Computers

- Antikythera Mechanism

(-87, authors?)

- Slide Rule



(1620 - 1630, Napier, Gunter, Wingate)

- Planimeter

(1814, Hermann)

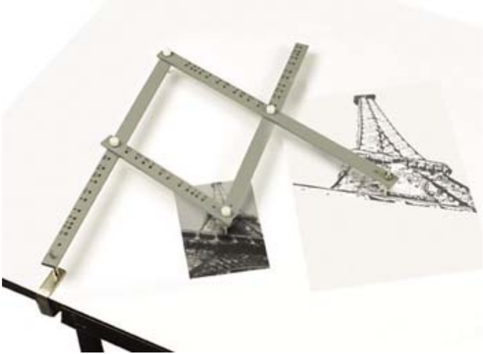
- MONIAC/Financephalograph

(1949, Phillips)

Sub-menu

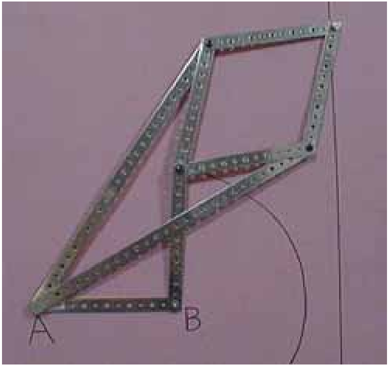
A model from 19th Century: Rivets' mechanisms

- Rivets' mechanisms.
  - How to realize an homothety: the pantograph.



A model from 19th Century: Rivets' mechanisms

- How to transform a circular into a linear motion: Peaucellier's mechanism (1864 - 1871).



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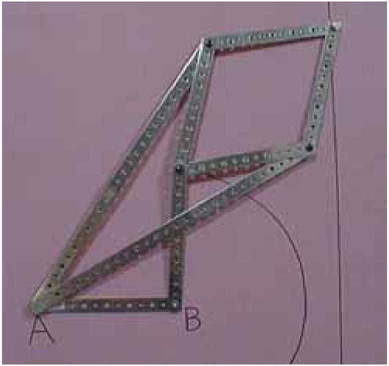

- Computational power of Rivets's mechanisms?

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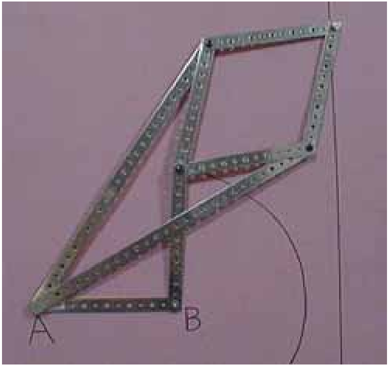

- Computational power of Rivets's mechanisms?
  - Theorem [Kempke]: computable iff semi-algebraic.

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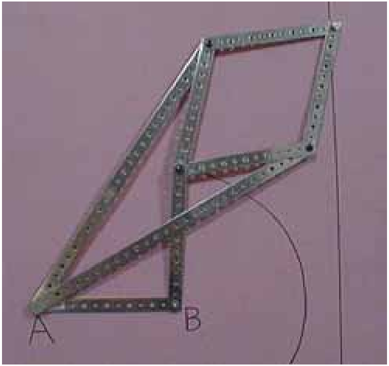


- Computational power of Rivets's mechanisms?
  - Theorem [Kempke]: computable iff semi-algebraic.

Voir aussi: "De la nécessité de tracer les droites au compas", Pierre Damphousse, Fête de la Science.

Formally

Theorem (Computational power of planar mechanisms)

- *For any non-empty semi-algebraic set S, there exists a mechanism with n points that move on linear segments, but that are free to move on these segments, and that forces the relation $(x_1, \ldots, x_n) \in S$, where x_i are the distances on the linear segments.*
- *Conversely, the domain of evolution of any finite planar mechanism is semi-algebraic.*

(theorem attributed to Kempke).

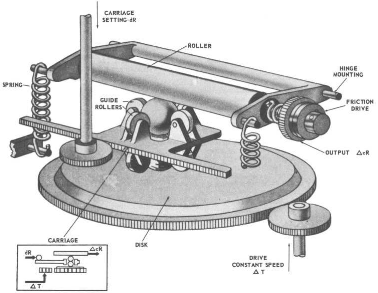
Sub-menu

A machine from 20th Century: Differential analyzers

Vannevar Bush's 1938 mechanical Differential Analyser

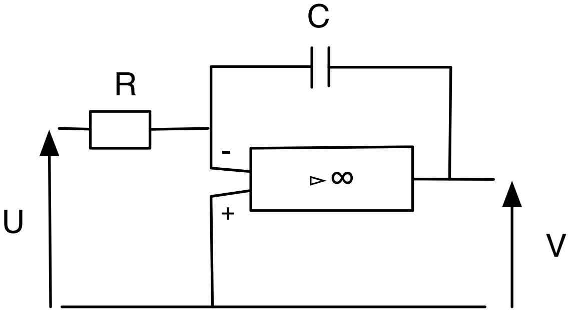
- Underlying principles: Lord Kelvin 1876.
- First ever built: V. Bush 1931 at MIT.
- Applications: from gunfire control up to aircraft design
- Intensively used during U.S. war effort.
- Electronic versions from late 40s, used until 70s

A Mechanical Integrator



Bureau of Naval Personnel, *Basic Machines and How They Work*, 1964

A Modern Electronic Integrator



$$V(t) = -1/RC \int_0^t U(t)dt$$

Electronic Differential Analyzer

Advertisements in Scientific American, March 1953.

Electronic Differential Analyzer

Advertisements in Scientific American, March 1953.

See also:

Doug Coward's
Analog Computer Museum

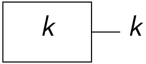
http://dcoward.best.vwh.net/analog/

The General Purpose Analog Computer

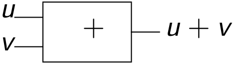
The GPAC

A mathematical abstraction from Claude Shannon (1941) of the Differential Analyzers.

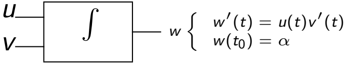
- Basic units:



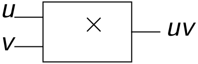
A constant unit



An adder unit

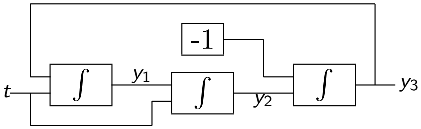


An integrator unit



A multiplier unit

Example: Generating cos and sin via a GPAC



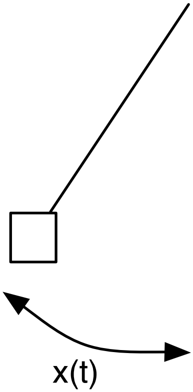
$$\begin{cases} y_1' = y_3 & \& \quad y_1(0) = 1 \\ y_2' = y_1 & \& \quad y_2(0) = 0 \\ y_3' = -y_2' & \& \quad y_3(0) = 0 \end{cases}$$

$$y_1 = \cos(t),\ y_2 = \sin(t),\ y_3 = -\sin(t).$$

Programming with GPACs: Example. Pendulum

Suppose you want to solve

$$x'' + p^2 \sin(x) = 0.$$



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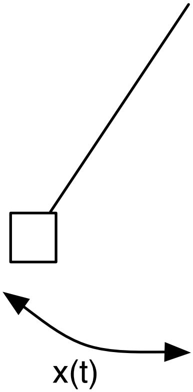

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Program:

Let's define

$$\begin{cases} y = x' \\ z = \sin(x) \\ u = \cos(x) \end{cases}$$

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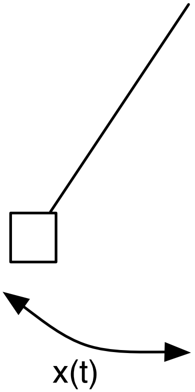

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Program:
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To get

$$\begin{cases} x' &= y \\ y' &= -p^2 z \\ z' &= yu \\ u' &= -yz \end{cases}.$$

Sub-menu

From discrete to continuous models

$$
\begin{array}{ll}
Q = \{q_1, q_2, q_3, q_4\} & \\
\delta(q_1, q_2) = q_2 & [\beta] \\
\delta(q_2, q_1) = q_2 & [\beta] \\
\delta(q_4, q_2) = q_3 : 1/2, q_4 : 1/2 & [\nu] \\
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\end{array}
$$

From discrete to continuous models

Microscopic Dynamic

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Macroscopic Dynamic

Kermack-McKendrick SIR model.

$$\left\{ \begin{array}{ccc} S' & = & -\beta SI \\ I' & = & \beta SI - \nu I \\ R' & = & \nu I. \end{array} \right.$$

From discrete to continuous models

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\right.
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Epidemic Rate

$$R = \beta S_0 / \nu.$$

From discrete to continuous models

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Epidemic Rate

$$R = \beta S_0 / \nu.$$

Question

Programming/Computing with Such models?

A model from 21th century: Computing with Large Populations

- **My favourite example:**
  - States: $\{{\color{red}\bullet},{\color{blue}\bullet}\}$
  - Rules of interactions:

$$
\begin{aligned}
{\color{red}\bullet}\ {\color{red}\bullet} &\rightarrow {\color{red}\bullet}\ {\color{blue}\bullet}\\
{\color{red}\bullet}\ {\color{blue}\bullet} &\rightarrow {\color{red}\bullet}\ {\color{red}\bullet}\\
{\color{blue}\bullet}\ {\color{red}\bullet} &\rightarrow {\color{red}\bullet}\ {\color{red}\bullet}\\
{\color{blue}\bullet}\ {\color{blue}\bullet} &\rightarrow {\color{red}\bullet}\ {\color{blue}\bullet}
\end{aligned}
$$

  - What can we say about

$$p_{\color{red}\bullet} = \frac{number\ of\ {\color{red}\bullet}}{number\ of\ {\color{red}\bullet} + number\ of\ {\color{blue}\bullet}}?$$

- **This is a model**
  - inspired from [Angluin,Aspnes,Diamadi,Fischer,Peralta 2004]'s Population Protocoles introduced in the context of distributed systems / (anonymous) sensor networks.
  - but with a large population hypothesis.

Informal approach on this example

$$\textcolor{red}{\bullet}\ \textcolor{red}{\bullet} \rightarrow \textcolor{red}{\bullet}\ \textcolor{blue}{\bullet}$$
$$\textcolor{red}{\bullet}\ \textcolor{blue}{\bullet} \rightarrow \textcolor{red}{\bullet}\ \textcolor{red}{\bullet}$$
$$\textcolor{blue}{\bullet}\ \textcolor{red}{\bullet} \rightarrow \textcolor{red}{\bullet}\ \textcolor{red}{\bullet}$$
$$\textcolor{blue}{\bullet}\ \textcolor{blue}{\bullet} \rightarrow \textcolor{red}{\bullet}\ \textcolor{blue}{\bullet}$$

- The mean number of $\textcolor{red}{\bullet}$ created,

$$\begin{aligned} b(p_{\textcolor{red}{\bullet}}) &= -1 * p_{\textcolor{red}{\bullet}}^2 + 1 * p_{\textcolor{red}{\bullet}}(1 - p_{\textcolor{red}{\bullet}}) + 1 * p_{\textcolor{red}{\bullet}}(1 - p_{\textcolor{red}{\bullet}}) + 1 * (1 - p_{\textcolor{red}{\bullet}})^2 \\ &= 1 - 2p_{\textcolor{red}{\bullet}}^2 \end{aligned}$$

  must be equal, at the limit to 0,
  and hence

$$p_{\textcolor{red}{\bullet}} = \frac{\sqrt{2}}{2},$$

  at the limit.

- In other words,

  this protocol **computes** real number $\frac{\sqrt{2}}{2}$.

Main result

Theorem

ν is computable by an LPP

if and only if

$\nu \in [0, 1]$ is algebraic.

Menu

What can be generated by a GPAC?

- The purpose of Shannon's 41 paper is a characterization of GPAC generable functions.
- Shannon's 41 characterization is incomplete: Several problems, even about definitions, corrected by [PourEl-Richards74], [Lipshitz-Rubel87], [Graça-Costa03].
- For the better defined class considered in [Graça-Costa03].

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- These functions will be also called *pIVP* functions.

Formally:

- For the better defined class considered in [Graça-Costa03], a scalar function $f : \mathbb{R} \to \mathbb{R}$ is generated by a GPAC iff

$$f(t) = y_i(t)$$

for $y(t) \in \mathbb{R}^m$ solution of

$$\begin{cases} y' & = & p(t, y), \\ y(0) & = & x \end{cases} \tag{1}$$

where p is (a vector of) polynomials.

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Uncomputability (Ungenerability) Results

Consequence: A GPAC generated unary function $f : I \subset \mathbb{R} \to \mathbb{R}$ must be differentially algebraic (d.a.):

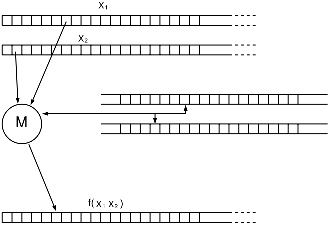
i.e. it satisfies some algebraic differential equation of the form $p\left(t, y, y', \ldots, y^{(n)}\right) = 0$, where p is a non-zero polynomial in all its variables.

Non-d.a. functions:

- Gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ [Hölder 1887].
- Riemann's Zeta function $\zeta(x) = \sum_{k=0}^\infty \frac{1}{k^x}$ [Hilbert].

Menu

Due to Turing, Grzegorczyk, Lacombe. Here presentation from Weihrauch.



A tape represents a real number

Each real number x is represented via an infinite sequence $(x_n)_n \in \mathbb{Q}$,

$$||x_n - x|| \leq 2^{-n}.$$

M behaves like a Turing Machine

Read-only one-way input tapes
Write-only one-way output tape.
M outputs a representation of $f(x_1, x_2)$ from representations of x_1, x_2.

Solving ODEs and computability

- Pour-El Richards 79:
  - There exists some computable $f : [0,1] \times [-1,1] \to \mathbb{R}$ such that ordinary differential equation

    $$y' = f(t, y),$$

    has no computable solution over any closed domain.

- Graça Zhong Buescu 2007:
  - If $f : [0,1] \times [-1,1] \to \mathbb{R}$ is computable and ordinary differential equation

    $$y' = f(t, y),$$

    has a **unique** solution, then it must be computable.

Moral on Analog Models of Computations

- **Summary:**

$$\text{GPAC generable} \subsetneq \text{Computable}$$

- **With more details:**

  - [Graça Zhong Buescu 2007] Let $f : (\alpha, \beta) \subset \mathbb{R} \to \mathbb{R}^k$ be some pIVP function with computable parameters.
    Then f is computable on (α, β).
  - pIVP functions must be analytic.
  - Computable functions include some non-analytic functions (ex: $\min(x, 0)$).
  - Gamma function and Riemann's Zeta function are computable.

Criticisms

We stated

$$\text{GPAC generable} \subsetneq \text{Computable}.$$

- However, the notion of GPAC generated function assumes computation in "real time" - a very restrictive form of computation.

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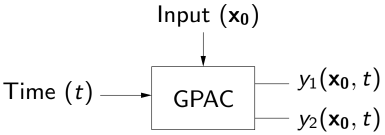
- However, the notion of GPAC generated function assumes computation in "real time" - a very restrictive form of computation.
- What happen if we change this notion of computability to the kind of "converging computation" used in recursive analysis,

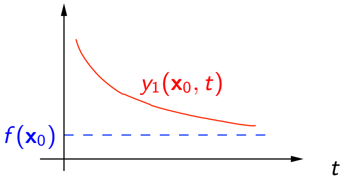
GPAC Computability vs GPAC Generation

Definition

A function $f : [a, b] \to \mathbb{R}$ is GPAC-computable iff there exist some computable polynomials $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $p_0 : \mathbb{R} \to \mathbb{R}$, and $n-1$ computable real values $\alpha_1, ..., \alpha_{n-1}$ such that:

1. $(y_1, ..., y_n)$ is the solution of the Cauchy problem $y' = p(y, t)$ with initial condition $(\alpha_1, ..., \alpha_{n-1}, p_0(x))$ set at time $t_0 = 0$
2. $\lim_{t\to\infty} y_2(t) = 0$
3. $|f(x) - y_1(t)| \leq y_2(t)$ for all $x \in [a, b]$ and all $t \in [0, +\infty)$.





Graça 04's Result

Proposition (Graça 04)

The Gamma function Γ *is GPAC-computable.*

(so is the ζ function)

Bournez, Campagnolo, Graça, Hainry's result

Theorem

Let a and b be computable reals. A function $f : [a, b] \to \mathbb{R}$ is computable iff it is GPAC-computable.

In a provocative way:

- GPAC is not weaker than modern machines, from a computability point of view.

Menu

An Other Morality & A BIG question?

- $TIME_{TM}(t) \subseteq TIME_{GPAC}(t)$.
- **Important question:**
  - Formulation 1: Can GPAC compute faster than Turing machines?
  - Formulation 2: $TIME_{GPAC}(t) \subseteq TIME_{TM}(t)$?

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- **Important question:**
  - Formulation 1: Can GPAC compute faster than Turing machines?
  - Formulation 2: $TIME_{GPAC}(t) \subseteq TIME_{TM}(t)$?
  - Formulation 3: Can (at least polynomial) ordinary differential equations be solved in polynomial time?

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Usual methods problems:

- Finite order method

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  $\Rightarrow$ Not polynomial

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- Unbounded order method
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- Do not assume Lipschitz functions

$\Rightarrow$ New problems !

A solution

We want to solve:

$$\begin{cases} y' &= p(y) \\ y(t_0) &= y_0 \end{cases}$$

A result (submitted):

- A simple algorithm: variable order multi-step Taylor method
- Tricky proof: error analysis and parameter choices

Main benefits: The proposed method is indeed polynomial !!

The algorithm

Algorithm 1: SolvePIVP

input : The initial condition $(t_0, y_0) \in \mathbb{Q} \times \mathbb{Q}^d$
input : The polynomial p of the PIVP
input : The total time step $T \in \mathbb{Q}$
input : The precision ξ requested
input : The number of steps N
input : The order of the method ω
output: $x \in \mathbb{Q}^d$

begin
  $\Delta \leftarrow \frac{T}{N}$
  $x \leftarrow y_0$
  for $n \leftarrow 1$ **to** N **do**
    $x \leftarrow \sum_{i=0}^{\omega-1} \frac{\Delta^i}{i!}\texttt{NthDeriv}(p, t_0 + n\Delta, x, \omega, \xi + \Delta)$

Main Result: Technical View

Theorem

Let $k = \deg(p)$, $\mu \geqslant 2$, $T \in \mathbb{Q}_+$, $Y \in \mathbb{Q}$ *such that*

$$Y \geqslant \sup_{t_0 \leqslant u \leqslant t_0 + T} \|y(u)\|_\infty$$

Then Previous Algorithm guarantees

$$\|y(t_0 + T) - \mathrm{SolvePIVP}(t_0, \tilde{y}_0, p, T, \omega, N, \omega)\|_\infty \leqslant e^{-\mu}$$

with the following parameters

$$\Delta = \frac{T}{N} \quad M = (2 + Y)^k \quad A = d(1 + k!\Sigma p M) \quad N = \lceil TeA \rceil$$

$$B = k4^k \Sigma p \Delta M \quad \omega = 2+\mu+\ln(N)+NB \quad \|y_0 - \tilde{y}_0\|_\infty \leqslant e^{-NB-\mu-1}$$

Morality on Analog Computations

- **Morality:** $TIME_{GPAC}(t) \subseteq TIME_{TM}(t)$ for pIVP functions that stay bounded or polynomially bounded.

$$Y \geqslant \sup_{t_0 \leqslant u \leqslant t_0+T} \|y(u)\|_\infty$$

- **Next question:** Is $TIME_{TM}(t) \subseteq TIME_{GPAC}(t)$ true for such functions.

- I.e. Derive a class of ODEs such that

$$Poly - Time_{TM} = Poly - Time_{ODE}.$$

Menu

Conclusions

- We saw various analog models of computations.
- No possible unification of all analog models:
  - Computability: Several models are provably different.
  - Complexity:
    - Even defining the time of a computation is problematic for some of them.

For the GPAC model

- **Important notice:** the GPAC is the only "general purpose" "physically motivated" model that we presented.

- **Analog Computability / Complexity:**
  - GPAC generable $\subsetneq$ Computable.
  - Computability: GPAC computable $=$ Computable.
  - Close to a notion of complexity for GPAC.

- **Promising perspective:**
  - Towards a complexity theory for analog models of computations.

The (digital) Picture

Church Thesis	"What is effectively calculable is computable"
Thesis M	"What can be calculated by a machine is computable"
Thesis?	"What can be calculated by a model is computable"

(following [Copeland2002])

Understanding computational power of models helps to understand

- limits of mechanical reasoning.
- limits of machines.
- limits of models.